Why Complex Numbers in Quantum Computing?

The Foundation of Quantum Mechanics

- **Quantum amplitudes:** State $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $\alpha, \beta \in \mathbb{C}$
- **Probability:** $P(0) = |\alpha|^2$, $P(1) = |\beta|^2$ with $|\alpha|^2 + |\beta|^2 = 1$
- **Phase matters:** $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ vs $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- **Unitary operations:** Quantum gates preserve normalization via complex rotations

Basic Operations

Core Complex Arithmetic

Let $z = a + bi$ where $i^2 = -1$

Operation	Formula
Addition	$(a+bi)+(c+di) = (a+c)+(b+d)i$
Multiplication	$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$
Conjugate	$\overline{a+bi} = a-bi$
Modulus	$\lvert a+bi\rvert = \sqrt{a^2+b^2}$
Division	$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{\lvert z_2\rvert^2}$

QC Example: For $\alpha = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, we have $|\alpha|^2 = \frac{1}{2} + \frac{1}{2} = 1$ (normalized)

Euler's Formula and Polar Form

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Key Values:

$e^{i0} = 1$	$e^{i\pi/2} = i$
$e^{i\pi} = -1$	$e^{i3\pi/2} = -i$
$e^{i2\pi} = 1$	$e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$

Polar Form

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

Conversions:

- $r = |z| = \sqrt{a^2+b^2}$
- $\theta = \arg(z) = \text{atan2}(b, a)$
- $a = r\cos\theta$, $b = r\sin\theta$

Multiplication in polar form: $z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$ (multiply magnitudes, add phases)

Quantum Gates as Complex Operations

Common Gate Phases

Gate	Action	Phase
X	Bit flip	Real (± 1)
Y	Bit + phase flip	$\pm i$
Z	Phase flip	± 1
S	$\sqrt{Z}$	$e^{i\pi/2} = i$
T	$\sqrt{S}$	$e^{i\pi/4}$
H	Superposition	$\frac{1}{\sqrt{2}}$

Bloch Sphere

General qubit state:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

- θ: polar angle (0 to π)
- ϕ: azimuthal angle (0 to 2π)
- Poles: $|0\rangle$ (north), $|1\rangle$ (south)
- Equator: equal superpositions

Essential Identities and Common Values

Powers of i

$$i^0 = 1$$
$$i^1 = i$$
$$i^2 = -1$$
$$i^3 = -i$$
$$i^4 = 1$$

Pattern: $i^n = i^{n \bmod 4}$

Key Properties

- $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$
- $z \cdot \overline{z} = |z|^2$
- $\overline{\overline{z}} = z$
- $(e^{i\theta})^* = e^{-i\theta}$

QC States

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$
$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$
$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

Quadratic Formula and Complex Solutions

When Real Numbers Aren't Enough

For $ax^2 + bx + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant $\Delta = b^2 - 4ac$:

- $\Delta > 0$: Two real solutions
- $\Delta = 0$: One real solution
- $\Delta < 0$: Two complex conjugate solutions

Example: $x^2 + x + 1 = 0$ gives $x = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
QC Connection: These complex roots represent rotations on the unit circle, similar to quantum phase gates

Complex Arithmetic in Quantum Circuits

Matrix Exponentials & Rotation Gates

Fundamental Formula: For Hermitian matrix H with $H^2 = I$:

$$e^{i\theta H} = \cos(\theta) I + i \sin(\theta) H$$

Pauli Rotation Gates:

$$R_x(\theta) = e^{-i\theta X/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_y(\theta) = e^{-i\theta Y/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_z(\theta) = e^{-i\theta Z/2} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

Special Values:

$$\begin{array}{l|l} R_x(\pi) = -iX & R_y(\pi) = -iY \\ R_z(\pi) = -iZ & R_z(\pi/2) = S \end{array}$$

Common Exponential Values:

Angle	Exponential
$e^{i\pi/8}$	$\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$
$e^{i\pi/4}$	$\frac{1+i}{\sqrt{2}}$
$e^{i\pi/3}$	$\frac{1+i\sqrt{3}}{2}$
$e^{i2\pi/3}$	$\frac{-1+i\sqrt{3}}{2}$

Useful Relations:

- $e^{i\theta} e^{-i\theta} = 1$
- $e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi}$
- $(e^{i\theta})^n = e^{in\theta}$
- $e^{-i\pi H} = -H$ for Pauli matrices
- $R_j(2\pi) = -I$, $R_j(4\pi) = I$