Chapter 37

Relativity

37-6 Momentum and Energy

In relativistic mechanics the definition of linear momentum is,

$$\vec{p} = \gamma m \vec{v} \quad \text{(momentum).}$$

This equation gives the correct definition of momentum for all physically possible speeds. For a speed much less than *c*, it reduces to the classical definition of momentum ($\boldsymbol{p} = m\boldsymbol{v}$) .

An object's mass *m* and the equivalent energy E_0 are related by

$$E_0 = mc^2,$$

which, without the subscript *0*, is the best-known science equation of all time. This energy E_0 that is associated with the mass of an object is called **mass energy** or **rest energy**. The second name suggests that E_0 is an energy that the object has even when it is at rest, simply because it has mass.
And if we assume that the object's potential energy is zero, then its total energy *E* is the sum of its mass energy and its kinetic energy:

$$E = E_0 + K = mc^2 + K.$$

Another expression for total energy *E* is $E = \gamma mc^2$

37-6 Momentum and Energy

An expression for kinetic energy that is correct for all physically possible speeds, including speeds close to c is given by

$$K = E - mc^2 = \gamma mc^2 - mc^2$$
$$= mc^2(\gamma - 1) \quad \text{(kinetic energy)},$$

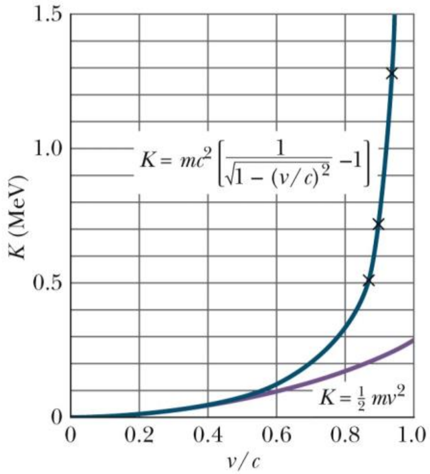
where $\gamma (= 1/\sqrt{1-(v/c)^2})$ is the Lorentz factor for the object's motion.

The connection between the relativistic momentum and kinetic energy is thus given by

$$(pc)^2 = K^2 + 2Kmc^2.$$

and

$$E^2 = (pc)^2 + (mc^2)^2.$$



The relativistic and classical equations for the kinetic energy of an electron, plotted as a function of v/c. Note that the two curves blend together at low speeds and diverge widely at high speeds. Experimental data (at the × marks) show that at high speeds the relativistic curve agrees with experiment but the classical curve does not.

Table 37-3 The Energy Equivalents of a Few Objects

Object	Mass (kg)	Energy Equivalent	
Electron	$\approx 9.11 \times 10^{-31}$	$\approx 8.19 \times 10^{-14}$ J	($\approx$ 511 keV)
Proton	$\approx 1.67 \times 10^{-27}$	$\approx 1.50 \times 10^{-10}$ J	($\approx$ 938 MeV)
Uranium atom	$\approx 3.95 \times 10^{-25}$	$\approx 3.55 \times 10^{-8}$ J	($\approx$ 225 GeV)
Dust particle	$\approx 1 \times 10^{-13}$	$\approx 1 \times 10^{4}$ J	($\approx$ 2 kcal)
U.S. penny	$\approx 3.1 \times 10^{-3}$	$\approx 2.8 \times 10^{14}$ J	($\approx$ 78 GW $\cdot$ h)

Checkpoint 4

Are (a) the kinetic energy and (b) the total energy of a 1 GeV electron more than, less than, or equal to those of a 1 GeV proton?

(a) equal; (b) less

Sample Problem 37.06 Energy and momentum of a relativistic electron

(a) What is the total energy E of a 2.53 MeV electron?

KEY IDEA

From Eq. 37-47, the total energy E is the sum of the electron's mass energy (or rest energy) mc^2 and its kinetic energy:

$$E = mc^2 + K. \tag{37-57}$$

Calculations: The adjective "2.53 MeV" in the problem statement means that the electron's kinetic energy is 2.53 MeV. To evaluate the electron's mass energy mc^2, we substitute the electron's mass m from Appendix B, obtaining

$$\begin{aligned} mc^2 &= (9.109 \times 10^{-31}\ \text{kg})(299\,792\,458\ \text{m/s})^2 \\ &= 8.187 \times 10^{-14}\ \text{J}. \end{aligned}$$

Then dividing this result by 1.602×10^{-13} J/MeV gives us 0.511 MeV as the electron's mass energy (confirming the value in Table 37-3). Equation 37-57 then yields

$$E = 0.511\ \text{MeV} + 2.53\ \text{MeV} = 3.04\ \text{MeV}. \quad \text{(Answer)}$$

(b) What is the magnitude p of the electron's momentum, in the unit MeV/c? (Note that c is the symbol for the speed of light and not itself a unit.)

KEY IDEA

We can find p from the total energy E and the mass energy mc^2 via Eq. 37-55,

$$E^2 = (pc)^2 + (mc^2)^2.$$

Calculations: Solving for pc gives us

$$\begin{aligned} pc &= \sqrt{E^2 - (mc^2)^2} \\ &= \sqrt{(3.04\ \text{MeV})^2 - (0.511\ \text{MeV})^2} = 3.00\ \text{MeV}. \end{aligned}$$

Finally, dividing both sides by c we find

$$p = 3.00\ \text{MeV}/c. \quad \text{(Answer)}$$

Sample Problem 37.07 Energy and an astounding discrepancy in travel time

The most energetic proton ever detected in the cosmic rays coming to Earth from space had an astounding kinetic energy of 3.0×10^{20} eV (enough energy to warm a teaspoon of water by a few degrees).

(a) What were the proton's Lorentz factor γ and speed v (both relative to the ground-based detector)?

KEY IDEAS

(1) The proton's Lorentz factor γ relates its total energy E to its mass energy mc^2 via Eq. 37-48 ($E = \gamma mc^2$). (2) The proton's total energy is the sum of its mass energy mc^2 and its (given) kinetic energy K.

Calculations: Putting these ideas together we have

$$\gamma = \frac{E}{mc^2} = \frac{mc^2 + K}{mc^2} = 1 + \frac{K}{mc^2}. \qquad (37\text{-}58)$$

From Table 37-3, the proton's mass energy mc^2 is 938 MeV. Substituting this and the given kinetic energy into Eq. 37-58, we obtain

$$\gamma = 1 + \frac{3.0 \times 10^{20}\ \text{eV}}{938 \times 10^6\ \text{eV}}$$

$$= 3.198 \times 10^{11} \approx 3.2 \times 10^{11}. \qquad \text{(Answer)}$$

This computed value for γ is so large that we cannot use the definition of γ (Eq. 37-8) to find v. Try it; your calculator will tell you that β is effectively equal to 1 and thus that v is effectively equal to c. Actually, v is almost c, but we want a more accurate answer, which we can obtain by first solving Eq. 37-8 for $1 - \beta$. To begin we write

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{(1-\beta)(1+\beta)}} \approx \frac{1}{\sqrt{2(1-\beta)}},$$

where we have used the fact that β is so close to unity that $1 + \beta$ is very close to 2. (We can round off the sum of two very close numbers but not their difference.) The velocity we seek is contained in the $1 - \beta$ term. Solving for $1 - \beta$ then yields

$$1 - \beta = \frac{1}{2\gamma^2} = \frac{1}{(2)(3.198 \times 10^{11})^2}$$

$$= 4.9 \times 10^{-24} \approx 5 \times 10^{-24}.$$

Thus, $$\beta = 1 - 5 \times 10^{-24}$$

and, since $v = \beta c$,

$$v \approx 0.999\,999\,999\,999\,999\,999\,999\,995c. \qquad \text{(Answer)}$$

(b) Suppose that the proton travels along a diameter of the Milky Way galaxy (9.8×10^4 ly). Approximately how long does the proton take to travel that diameter as measured from the common reference frame of Earth and the Galaxy?

Reasoning: We just saw that this *ultrarelativistic* proton is traveling at a speed barely less than c. By the definition of light-year, light takes 1 y to travel a distance of 1 ly, and so light should take 9.8×10^4 y to travel 9.8×10^4 ly, and this proton should take almost the same time. Thus, from our Earth–Milky Way reference frame, the proton's trip takes

$$\Delta t = 9.8 \times 10^4\ \text{y}. \qquad \text{(Answer)}$$

(c) How long does the trip take as measured in the reference frame of the proton?

KEY IDEAS

1. This problem involves measurements made from two (inertial) reference frames: one is the Earth–Milky Way frame and the other is attached to the proton.
2. This problem also involves two events: the first is when the proton passes one end of the diameter along the Galaxy, and the second is when it passes the opposite end.
3. The time interval between those two events as measured in the proton's reference frame is the proper time interval Δt_0 because the events occur at the same location in that frame—namely, at the proton itself.
4. We can find the proper time interval Δt_0 from the time interval Δt measured in the Earth–Milky Way frame by using Eq. 37-9 ($\Delta t = \gamma\, \Delta t_0$) for time dilation. (Note that we can use that equation because one of the time measures *is* a proper time. However, we get the same relation if we use a Lorentz transformation.)

Calculation: Solving Eq. 37-9 for Δt_0 and substituting γ from (a) and Δt from (b), we find

$$\Delta t_0 = \frac{\Delta t}{\gamma} = \frac{9.8 \times 10^4 \text{ y}}{3.198 \times 10^{11}}$$

$$= 3.06 \times 10^{-7} \text{ y} = 9.7 \text{ s.} \qquad \text{(Answer)}$$

In our frame, the trip takes 98 000 y. In the proton's frame, it takes 9.7 s! As promised at the start of this chapter, relative motion can alter the rate at which time passes, and we have here an extreme example.

37 Summary

The Postulates

- Einstein's special theory of relativity is based on two postulates:

1. The laws of physics are the same for observers in all inertial reference frames. No one frame is preferred over any other.
2. The speed of light in vacuum has the same value *c* in all directions and in all inertial reference frames.

Time Dilation

- For an observer moving with relative speed *v*, the measured time interval is

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-(v/c)^2}} = \frac{\Delta t_0}{\sqrt{1-\beta^2}} = \gamma\,\Delta t_0 \quad \text{(time dilation)}.$$

Eq. 37-7 to 9

Length Contraction

- For an observer moving with relative speed v, the measured length is

$$L = L_0\sqrt{1-\beta^2} = \frac{L_0}{\gamma}$$

Eq. 37-13

The Lorentz Transformation

- The Lorentz transformation equations relate the space time coordinates of a single event as seen by observers in two inertial frames and are given by

$$x' = \gamma(x - vt),$$
$$y' = y,$$
$$z' = z,$$
$$t' = \gamma(t - vx/c^2).$$

Eq. 37-21

37 Summary

Relativity of Velocities

- Relativistic addition of velocities is given by

$$u = \frac{u' + v}{1 + u'v/c^2}$$ **Eq. 37-29**

Relativistic Doppler Effect

- When the separation between the detector and the light source is increasing, the wavelengths are related by

$$\lambda = \lambda_0 \sqrt{\frac{1 + \beta}{1 - \beta}}$$ **Eq. 37-32**

- For speeds much less than c, the magnitude of the Doppler wavelength shift is approximately related to v by

$$v = \frac{|\Delta\lambda|}{\lambda_0} c \quad (v \ll c).$$ **Eq. 37-36**

Transverse Doppler Effect

- If the relative motion of the light source is perpendicular to a line joining the source and detector, then

$$f = f_0 \sqrt{1 - \beta^2}.$$ **Eq. 37-37**

Momentum and Energy

- The following definitions of linear momentum ***p***, kinetic energy *K*, and total energy *E* for a particle of mass *m* are valid at any physically possible speed:

$$\vec{p} = \gamma m \vec{v}$$ **Eq. 37-42**

$$E = mc^2 + K = \gamma mc^2$$ **Eq. 37-47&48**

$$K = mc^2(\gamma - 1)$$ **Eq. 37-52**

- These equations lead to the relationships

$$(pc)^2 = K^2 + 2Kmc^2$$ **Eq. 37-54**

$$E^2 = (pc)^2 + (mc^2)^2.$$ **Eq. 37-55**