Chapter 38

Photons and Matter Waves

38.7: Heisenberg's Uncertainty Principle

Our inability to predict the position of a particle with a uniform electric potential energy, as indicated by Fig. 38-13, is our first example of **Heisenberg's uncertainty principle,** proposed in 1927 by German physicist Werner Heisenberg. It states that measured values cannot be assigned to the position $\vec{r}$ and the momentum $\vec{p}$ of a particle simultaneously with unlimited precision.

In terms of $\hbar = h/2\pi$ (called "h-bar"), the principle tells us

$$\boxed{\begin{aligned} \Delta x \cdot \Delta p_x &\geq \hbar \\ \Delta y \cdot \Delta p_y &\geq \hbar \qquad \text{(Heisenberg's uncertainty principle).} \\ \Delta z \cdot \Delta p_z &\geq \hbar \end{aligned}}$$

Heisenberg's Uncertainty Principle states that measured values cannot be assigned to the position and the momentum of a particle simultaneously with unlimited precision.

Here Δx and Δp_x represent the intrinsic uncertainties in the measurements of the x components of ***r*** and ***p***, with parallel meanings for the y and z terms. Even with the best measuring instruments, each product of a position uncertainty and a momentum uncertainty will be greater than $\hbar$, never less.

Example, Uncertainty Principle, position and momentum:

Assume that an electron is moving along an x axis and that you measure its speed to be 2.05×10^6 m/s, which can be known with a precision of 0.50%. What is the minimum uncertainty (as allowed by the uncertainty principle in quantum theory) with which you can simultaneously measure the position of the electron along the x axis?

KEY IDEA

The minimum uncertainty allowed by quantum theory is given by Heisenberg's uncertainty principle in Eq. 38-20. We need only consider components along the x axis because we have motion only along that axis and want the uncertainty Δx in location along that axis. Since we want the minimum allowed uncertainty, we use the equality instead of the inequality in the x-axis part of Eq. 38-20, writing $\Delta x \cdot \Delta p_x = \hbar$.

Calculations: To evaluate the uncertainty Δp_x in the momentum, we must first evaluate the momentum component p_x. Because the electron's speed v_x is much less than the speed of light c, we can evaluate p_x with the classical expression for momentum instead of using a relativistic expression. We find

$$\begin{aligned} p_x = mv_x &= (9.11 \times 10^{-31}\ \text{kg})(2.05 \times 10^6\ \text{m/s}) \\ &= 1.87 \times 10^{-24}\ \text{kg}\cdot\text{m/s}. \end{aligned}$$

The uncertainty in the speed is given as 0.50% of the measured speed. Because p_x depends directly on speed, the uncertainty Δp_x in the momentum must be 0.50% of the momentum:

$$\begin{aligned} \Delta p_x &= (0.0050)p_x \\ &= (0.0050)(1.87 \times 10^{-24}\ \text{kg}\cdot\text{m/s}) \\ &= 9.35 \times 10^{-27}\ \text{kg}\cdot\text{m/s}. \end{aligned}$$

Then the uncertainty principle gives us

$$\begin{aligned} \Delta x = \frac{\hbar}{\Delta p_x} &= \frac{(6.63 \times 10^{-34}\ \text{J}\cdot\text{s})/2\pi}{9.35 \times 10^{-27}\ \text{kg}\cdot\text{m/s}} \\ &= 1.13 \times 10^{-8}\ \text{m} \approx 11\ \text{nm}, \qquad \text{(Answer)} \end{aligned}$$

which is about 100 atomic diameters. Given your measurement of the electron's speed, it makes no sense to try to pin down the electron's position to any greater precision.

38.8: Reflection from a Potential Step

In Fig. 38-14, we send a beam of a great many nonrelativistic electrons, each of total energy E, along an x axis through a narrow tube. Initially they are in region 1 where their potential energy is $U = 0$, but at $x = 0$ they encounter a region with a negative electric potential V_b. The transition is called a *potential step* or *potential energy step*. The step is said to have a *height* U_b, which is the potential energy an electron will have once it passes through the boundary at $x = 0$, as plotted in Fig. 38-15 for potential energy as a function of position x. (Recall that $U = qV$. Here the potential V_b is negative, the electron's charge q is negative, and so the potential energy U_b is positive.)

Classically, the electron has too much energy to be reflected by the potential step.

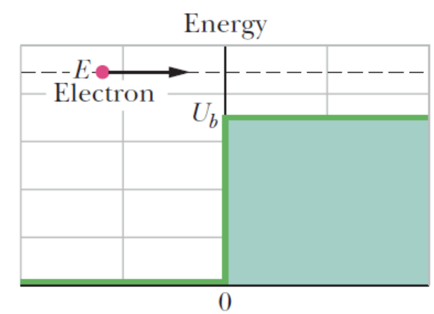


Figure 38-15 An energy diagram containing two plots for the situation of Fig. 38-14: (1) The electron's mechanical energy E is plotted. (2) The electron's electric potential energy U is plotted as a function of the electron's position x. The nonzero part of the plot (the potential step) has height U_b.

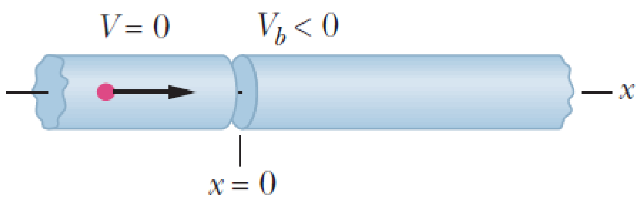


Figure 38-14 The elements of a tube in which an electron (the dot) approaches a region with a negative electric potential V_b.

According to classical physics, if a particle's initial kinetic energy exceeds the potential energy, it should never be reflected by the region. However, according to quantum physics, there is a reflection coefficient R that gives a finite probability of reflection. The probability of transmission is $T = 1 - R$.

38.9: Tunneling Through a Potential Barrier:

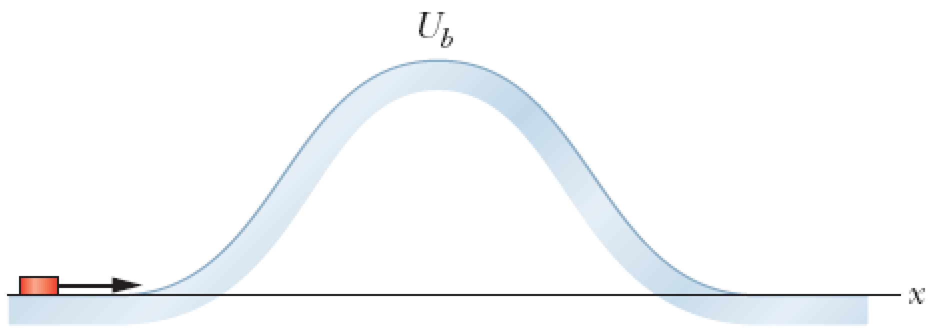


Fig. 38-13 A puck slides over frictionless ice toward a hill. The puck's gravitational potential at the top of the hill will be U_b.

As the puck climbs the hill, kinetic energy K is transformed into gravitational potential energy U. If the puck reaches the top, its potential energy is U_b. Thus, the puck can pass over the top only if its initial mechanical energy $E > U_b$.

The hill acts as a ***potential energy barrier*** (or, for short, ***a potential barrier).***

There is a potential barrier for a nonrelativistic electron traveling along an idealized wire of negligible thickness (Figure 38-14). The electron, with mechanical energy *E*, approaches a region (the barrier) in which the electric potential V_b is negative.

The electron, being negatively charged, will have a positive potential energy U_b $(=qV_b)$ in that region (Fig. 38-15). If $E > U_b$, we expect the electron to pass through the barrier region and come out to the right of $x = L$ in Fig. 38-14. If $E < U_b$, we expect the electron to be unable to pass through the barrier region.

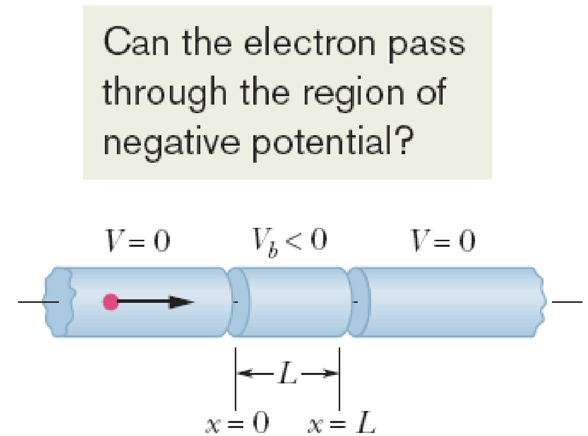


Fig. 38-14 The elements of an idealized thin wire in which an electron (the dot) approaches a negative electric potential V_b in the region $x = 0$ to $x = L$.

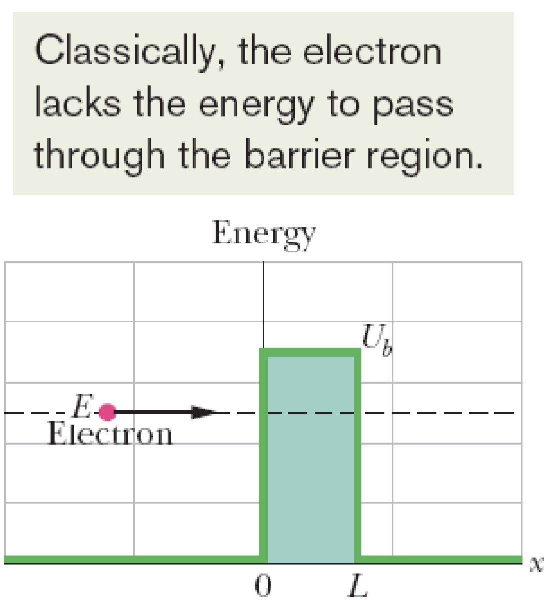


Fig. 38-15 An electron's mechanical energy *E* is plotted when the electron is at any coordinate $x<0$.
The electron's electric potential energy *U* is plotted as a function of the electron's position *x*, assuming that the electron can reach any value of *x*. The nonzero part of the plot (the potential barrier) has height U_b and thickness *L*.

Something astounding can happen to the electron when $E < U_b$.

Since it is a matter wave, the electron has a finite probability of leaking (or, *tunneling)* through the barrier and materializing on the other side, moving rightward with energy E as though nothing had happened in the region of $0 \le x \le L$.

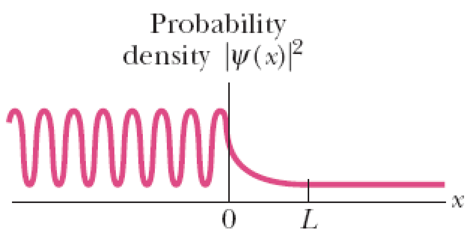
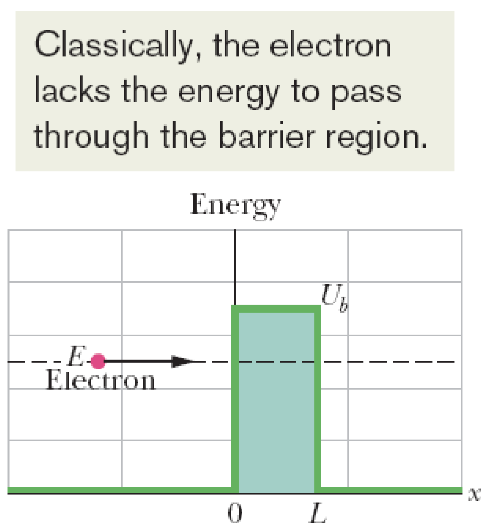


Fig. 38-16 A plot of the probability density $|\psi|^2$ of the electron matter wave for the situation of Fig. 38-15. The value of $|\psi|^2$ is nonzero to the right of the potential barrier.



The wave function $\psi(x)$ describing the electron can be found by solving Schrödinger's equation separately for the three regions: (1) to the left of the barrier, (2) within the barrier, and (3) to the right of the barrier.

The arbitrary constants that appear in the solutions can then be chosen so that the values of $\psi(x)$ and its derivative with respect to x join smoothly at $x = 0$ and at $x = L$. Squaring the absolute value of $\psi(x)$ then yields the probability density.

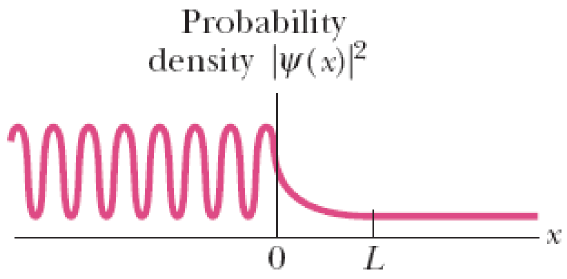


Fig. 38-16 A plot of the probability density $|\psi|^2$ of the electron matter wave for the situation of Fig. 38-15. The value of $|\psi|^2$ is nonzero to the right of the potential barrier.

- Within the barrier the probability density decreases exponentially with *x.*
- To the right of the barrier, the probability density plot describes a transmitted (through the barrier) wave with low but constant amplitude.

- We can assign a ***transmission coefficient T*** to the incident matter wave and the barrier. This coefficient gives the probability with which an approaching electron will be transmitted through the barrier—that is, that tunneling will occur. For example if T = 0.020, then for every 1000 electrons fired at the barrier, 20 will tunnel through and 980 will be reflected back.

Approximately,

$$T \approx e^{-2bL},$$

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}},$$

the value of T is very sensitive to the three variables on which it depends: particle mass m, barrier thickness L, and energy difference $U_b - E$. (Because we do not include relativistic effects here, E does not include mass energy.)

Barrier Tunneling, The Scanning Tunneling Microscope (STM):

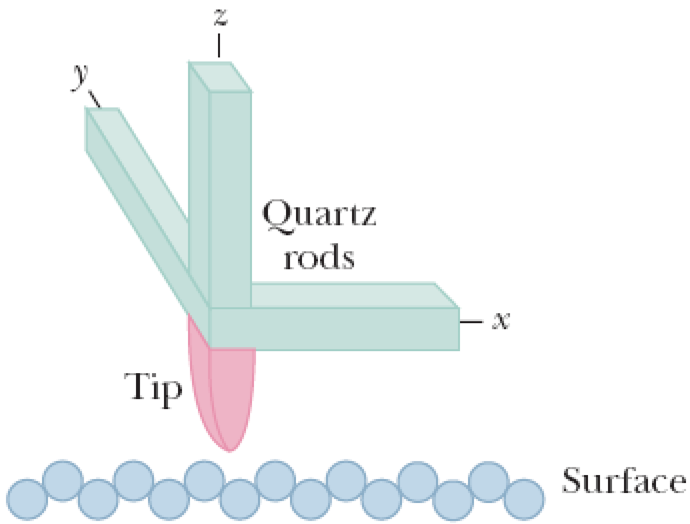


Fig. 38-17 The essence of a scanning tunneling microscope (STM). Three quartz rods are used to scan a sharply pointed conducting tip across the surface of interest and to maintain a constant separation between tip and surface. The tip thus moves up and down to match the contours of the surface, and a record of its movement provides information for a computer to create an image of the surface.

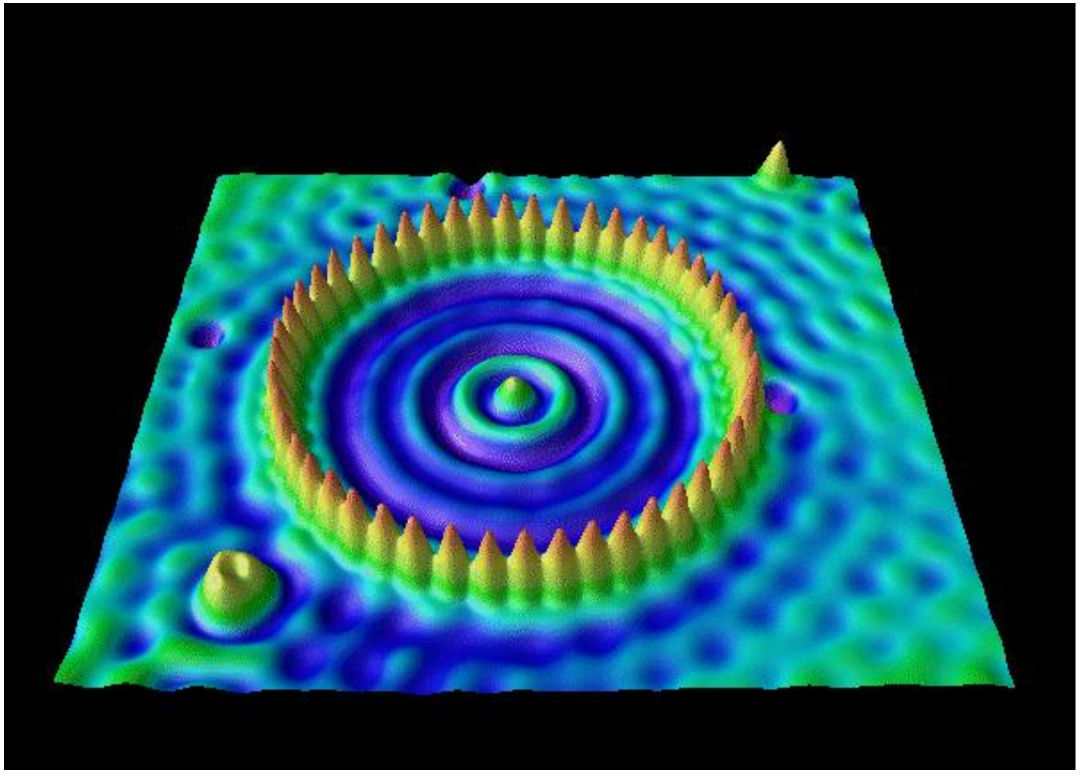
The Nobel Prize in Physics 1986

Ernst Ruska
Prize share: 1/2

Gerd Binnig
Prize share: 1/4

Heinrich Rohrer
Prize share: 1/4

The Nobel Prize in Physics 1986 was divided, one half awarded to Ernst Ruska *"for his fundamental work in electron optics, and for the design of the first electron microscope"*, the other half jointly to Gerd Binnig and Heinrich Rohrer *"for their design of the scanning tunneling microscope"*.



Iron atoms on the surface of Cu(111)

Sample Problem 38.06 Barrier tunneling by matter wave

Suppose that the electron in Fig. 38-17, having a total energy E of 5.1 eV, approaches a barrier of height $U_b = 6.8$ eV and thickness $L = 750$ pm.

(a) What is the approximate probability that the electron will be transmitted through the barrier, to appear (and be detectable) on the other side of the barrier?

KEY IDEA

The probability we seek is the transmission coefficient T as given by Eq. 38-38 ($T \approx e^{-2bL}$), where

$$b = \sqrt{\frac{8\pi^2 m(U_b - E)}{h^2}}.$$

Calculations: The numerator of the fraction under the square-root sign is

$$(8\pi^2)(9.11 \times 10^{-31}\text{ kg})(6.8\text{ eV} - 5.1\text{ eV}) \times (1.60 \times 10^{-19}\text{ J/eV}) = 1.956 \times 10^{-47}\text{ J}\cdot\text{kg}.$$

Thus, $$b = \sqrt{\frac{1.956 \times 10^{-47}\text{ J}\cdot\text{kg}}{(6.63 \times 10^{-34}\text{ J}\cdot\text{s})^2}} = 6.67 \times 10^{9}\text{ m}^{-1}.$$

The (dimensionless) quantity $2bL$ is then

$$2bL = (2)(6.67 \times 10^{9}\text{ m}^{-1})(750 \times 10^{-12}\text{ m}) = 10.0$$

and, from Eq. 38-38, the transmission coefficient is

$$T \approx e^{-2bL} = e^{-10.0} = 45 \times 10^{-6}. \qquad \text{(Answer)}$$

Thus, of every million electrons that strike the barrier, about 45 will tunnel through it, each appearing on the other side with its original total energy of 5.1 eV. (The transmission through the barrier does not alter an electron's energy or any other property.)

(b) What is the approximate probability that a proton with the same total energy of 5.1 eV will be transmitted through the barrier, to appear (and be detectable) on the other side of the barrier?

Reasoning: The transmission coefficient T (and thus the probability of transmission) depends on the mass of the particle. Indeed, because mass m is one of the factors in the exponent of e in the equation for T, the probability of transmission is very sensitive to the mass of the particle. This time, the mass is that of a proton (1.67×10^{-27} kg), which is significantly greater than that of the electron in (a). By substituting the proton's mass for the mass in (a) and then continuing as we did there, we find that $T \approx 10^{-186}$. Thus, although the probability that the proton will be transmitted is not exactly zero, it is barely more than zero. For even more massive particles with the same total energy of 5.1 eV, the probability of transmission is exponentially lower.