CH31: Electromagnetic Oscillations and Alternating Current
Lecture 7

31-4 The Series RLC Circuit

For a series RLC circuit with an external emf given by

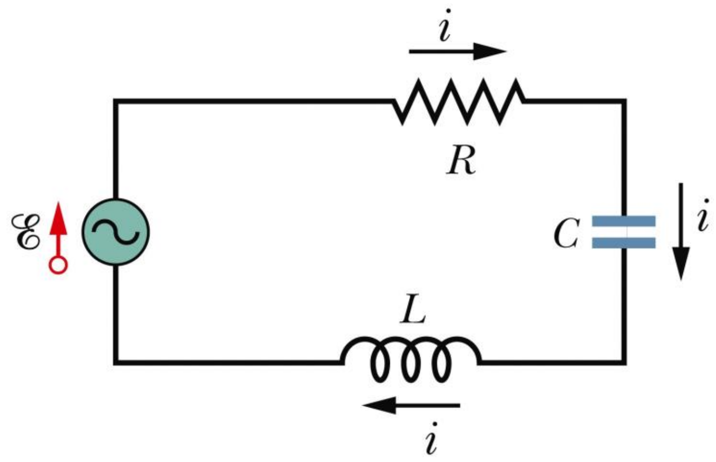
$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t$$

The current is given by

$$i = I \sin(\omega_d t - \phi)$$

the current amplitude is given by

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}}.$$



Series RLC circuit with an external emf

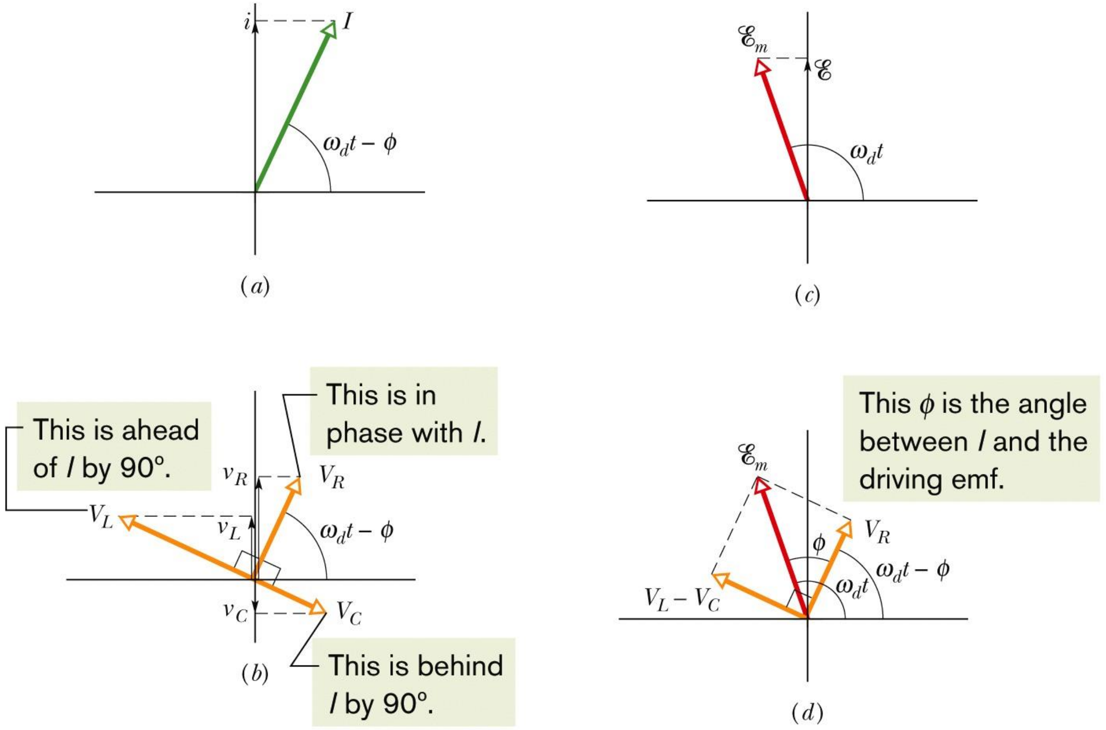
The denominator in the above equation is called the impedance Z of the circuit for the driving angular frequency ω_d.

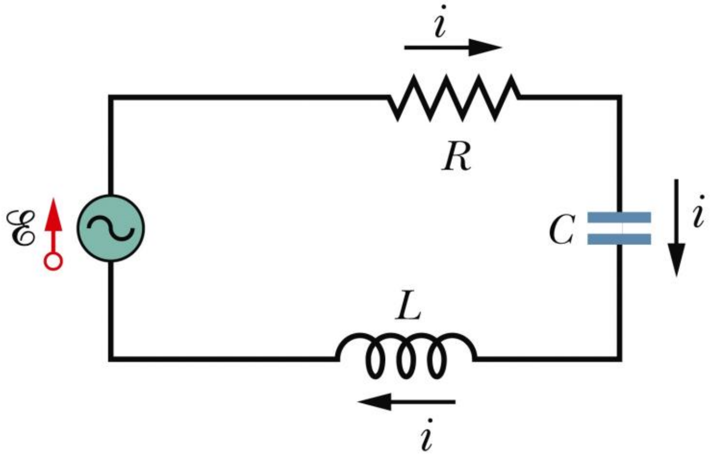
$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

If we substitute the value of X_L and X_C in the equation for current (I), the equation becomes:

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}}$$

31-4 The Series RLC Circuits





Series RLC circuit
with an external emf

From the right-hand phasor triangle in Fig.(d) we can write

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR},$$

$$\tan\phi = \frac{X_L - X_C}{R}$$

Phase Constant

The current amplitude I is maximum when the driving angular frequency ω_d equals the natural angular frequency ω of the circuit, a condition known as **resonance**. Then $X_C = X_L$, $\phi = 0$, and the current is in phase with the emf.

$$\omega_d = \omega = \frac{1}{\sqrt{LC}} \quad \text{(resonance)}.$$

Checkpoint 6

Here are the capacitive reactance and inductive reactance, respectively, for three sinusoidally driven series *RLC* circuits: (1) 50 Ω, 100 Ω; (2) 100 Ω, 50 Ω; (3) 50 Ω, 50 Ω. (a) For each, does the current lead or lag the applied emf, or are the two in phase? (b) Which circuit is in resonance?

(a) 1, lags; 2, leads; 3, in phase; (b) 3 ($\omega_d = \omega$ when $X_L = X_C$)

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{IX_L - IX_C}{IR},$$

Sample Problem 31.06 Current amplitude, impedance, and phase constant

In Fig. 31-7, let $R = 200\ \Omega$, $C = 15.0\ \mu\text{F}$, $L = 230$ mH, $f_d = 60.0$ Hz, and $\mathscr{E}_m = 36.0$ V. (These parameters are those used in the earlier sample problems.)

(a) What is the current amplitude I?

KEY IDEA

The current amplitude I depends on the amplitude $\mathscr{E}_m$ of the driving emf and on the impedance Z of the circuit, according to Eq. 31-62 ($I = \mathscr{E}_m/Z$).

Calculations: So, we need to find Z, which depends on resistance R, capacitive reactance X_C, and inductive reactance X_L. The circuit's resistance is the given resistance R. Its capacitive reactance is due to the given capacitance and, from an earlier sample problem, $X_C = 177\ \Omega$. Its inductive reactance is due to the given inductance and, from another sample problem, $X_L = 86.7\ \Omega$. Thus, the circuit's impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$
$$= \sqrt{(200\ \Omega)^2 + (86.7\ \Omega - 177\ \Omega)^2}$$
$$= 219\ \Omega.$$

We then find

$$I = \frac{\mathscr{E}_m}{Z} = \frac{36.0\ \text{V}}{219\ \Omega} = 0.164\ \text{A}. \qquad \text{(Answer)}$$

(b) What is the phase constant ϕ of the current in the circuit relative to the driving emf?

KEY IDEA

The phase constant depends on the inductive reactance, the capacitive reactance, and the resistance of the circuit, according to Eq. 31-65.

Calculation: Solving Eq. 31-65 for ϕ leads to

$$\phi = \tan^{-1}\frac{X_L - X_C}{R} = \tan^{-1}\frac{86.7\ \Omega - 177\ \Omega}{200\ \Omega}$$
$$= -24.3° = -0.424\ \text{rad}. \qquad \text{(Answer)}$$

The negative phase constant is consistent with the fact that the load is mainly capacitive; that is, $X_C > X_L$. In the common mnemonic for driven series RLC circuits, this circuit is an *ICE* circuit—the current *leads* the driving emf.

31-5 Power in Alternating-Current Circuits

The instantaneous rate at which energy is dissipated in the resistor can be written as

$$P = i^2R = [I\sin(\omega_d t - \phi)]^2R = I^2R\sin^2(\omega_d t - \phi).$$

Over one complete cycle, the average value of $\sin\theta$, where θ is any variable, is zero (Fig.a) but the average value of $\sin^2\theta$ is 1/2(Fig.b). Thus the power is,

$$P_{avg} = \frac{I^2R}{2} = \left(\frac{I}{\sqrt{2}}\right)^2 R.$$

The quantity $I/\sqrt{2}$ is called the **root-mean-square**, or rms, value of the current i:

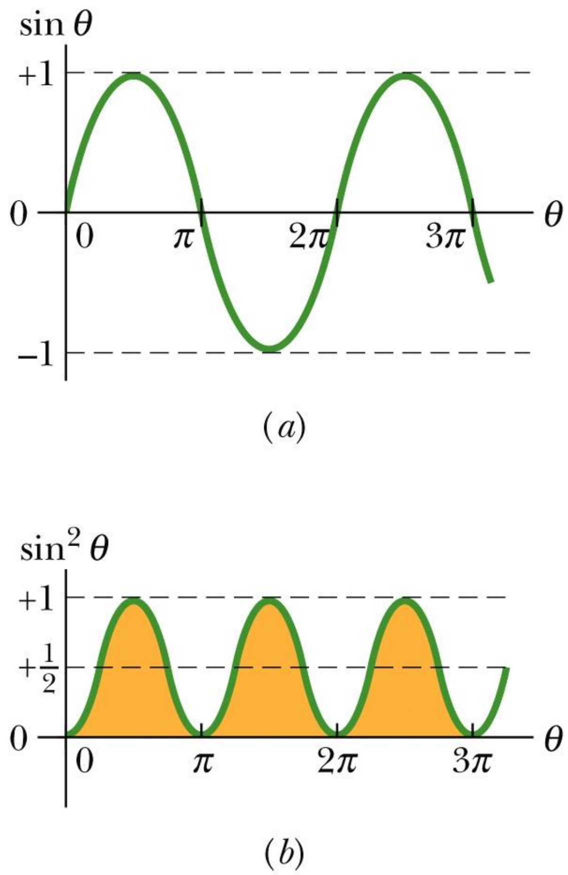
$$I_{rms} = \frac{I}{\sqrt{2}} \quad \Rightarrow \quad P_{avg} = I_{rms}^2 R$$

We can also define rms values of voltages and emfs for alternating-current circuits:

$$V_{rms} = \frac{V}{\sqrt{2}} \quad \text{and} \quad \mathscr{E}_{rms} = \frac{\mathscr{E}_m}{\sqrt{2}}$$

In a series RLC circuit, the average power P_{avg} of the generator is equal to the production rate of thermal energy in the resistor:

$$P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi \quad \text{(average power)},$$



(a) A plot of $\sin\theta$ versus θ. The average value over one cycle is zero.
(b) A plot of $\sin^2\theta$ versus θ. The average value over one cycle is 1/2.

Sample Problem 31.07 Driven *RLC* circuit: power factor and average power

A series *RLC* circuit, driven with $\mathscr{E}_{rms} = 120$ V at frequency $f_d = 60.0$ Hz, contains a resistance $R = 200\ \Omega$, an inductance with inductive reactance $X_L = 80.0\ \Omega$, and a capacitance with capacitive reactance $X_C = 150\ \Omega$.

(a) What are the power factor $\cos\phi$ and phase constant ϕ of the circuit?

KEY IDEA

The power factor $\cos\phi$ can be found from the resistance R and impedance Z via Eq. 31-75 ($\cos\phi = R/Z$).

Calculations: To calculate Z, we use Eq. 31-61:

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$
$$= \sqrt{(200\ \Omega)^2 + (80.0\ \Omega - 150\ \Omega)^2} = 211.90\ \Omega.$$

Equation 31-75 then gives us

$$\cos\phi = \frac{R}{Z} = \frac{200\ \Omega}{211.90\ \Omega} = 0.9438 \approx 0.944. \qquad \text{(Answer)}$$

Taking the inverse cosine then yields

$$\phi = \cos^{-1} 0.944 = \pm 19.3°.$$

The inverse cosine on a calculator gives only the positive answer here, but both $+19.3°$ and $-19.3°$ have a cosine of 0.944. To determine which sign is correct, we must consider whether the current leads or lags the driving emf. Because $X_C > X_L$, this circuit is mainly capacitive, with the current leading the emf. Thus, ϕ must be negative:

$$\phi = -19.3°. \qquad \text{(Answer)}$$

We could, instead, have found ϕ with Eq. 31-65. A calculator would then have given us the answer with the minus sign.

(b) What is the average rate P_{avg} at which energy is dissipated in the resistance?

KEY IDEAS

There are two ways and two ideas to use: (1) Because the circuit is assumed to be in steady-state operation, the rate at which energy is dissipated in the resistance is equal to the rate at which energy is supplied to the circuit, as given by Eq. 31-76 ($P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi$). (2) The rate at which energy is dissipated in a resistance R depends on the square of the rms current I_{rms} through it, according to Eq. 31-71 ($P_{avg} = I^2_{rms} R$).

First way: We are given the rms driving emf $\mathscr{E}_{rms}$ and we already know $\cos\phi$ from part (a). The rms current I_{rms} is determined by the rms value of the driving emf and the circuit's impedance Z (which we know), according to Eq. 31-73:

$$I_{rms} = \frac{\mathscr{E}_{rms}}{Z}.$$

Substituting this into Eq. 31-76 then leads to

$$P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi = \frac{\mathscr{E}^2_{rms}}{Z} \cos\phi$$
$$= \frac{(120\ \text{V})^2}{211.90\ \Omega}(0.9438) = 64.1\ \text{W}. \qquad \text{(Answer)}$$

Second way: Instead, we can write

$$P_{avg} = I^2_{rms} R = \frac{\mathscr{E}^2_{rms}}{Z^2} R$$
$$= \frac{(120\ \text{V})^2}{(211.90\ \Omega)^2}(200\ \Omega) = 64.1\ \text{W}. \qquad \text{(Answer)}$$

(c) What new capacitance C_{new} is needed to maximize P_{avg} if the other parameters of the circuit are not changed?

KEY IDEAS

(1) The average rate P_{avg} at which energy is supplied and dissipated is maximized if the circuit is brought into resonance with the driving emf. (2) Resonance occurs when $X_C = X_L$.

Calculations: From the given data, we have $X_C > X_L$. Thus, we must decrease X_C to reach resonance. From Eq. 31-39 ($X_C = 1/\omega_d C$), we see that this means we must increase C to the new value C_{new}.

Using Eq. 31-39, we can write the resonance condition $X_C = X_L$ as

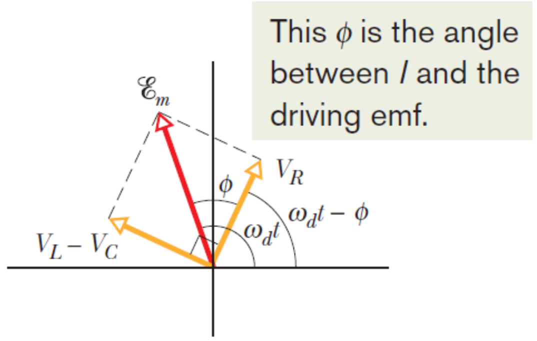
$$\frac{1}{\omega_d C_{new}} = X_L.$$

Substituting $2\pi f_d$ for ω_d (because we are given f_d and not ω_d) and then solving for C_{new}, we find

$$C_{new} = \frac{1}{2\pi f_d X_L} = \frac{1}{(2\pi)(60\ \text{Hz})(80.0\ \Omega)}$$
$$= 3.32 \times 10^{-5}\ \text{F} = 33.2\ \mu\text{F}. \qquad \text{(Answer)}$$

Following the procedure of part (b), you can show that with C_{new}, the average power of energy dissipation P_{avg} would then be at its maximum value of

$$P_{avg,\,max} = 72.0\ \text{W}.$$



(*d*)

$$I_{rms} = \frac{\mathscr{E}_{rms}}{Z} = \frac{\mathscr{E}_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}}, \qquad (31\text{-}73)$$

and, indeed, this is the form that we almost always use.

We can use the relationship $I_{rms} = \mathscr{E}_{rms}/Z$ to recast Eq. 31-71 in a useful equivalent way. We write

$$P_{avg} = \frac{\mathscr{E}_{rms}}{Z} I_{rms} R = \mathscr{E}_{rms} I_{rms} \frac{R}{Z}. \qquad (31\text{-}74)$$

From Fig. 31-14*d*, Table 31-2, and Eq. 31-62, however, we see that R/Z is just the cosine of the phase constant ϕ:

$$\cos\phi = \frac{V_R}{\mathscr{E}_m} = \frac{IR}{IZ} = \frac{R}{Z}. \qquad (31\text{-}75)$$

Equation 31-74 then becomes

$$P_{avg} = \mathscr{E}_{rms} I_{rms} \cos\phi \quad \text{(average power)}, \qquad (31\text{-}76)$$

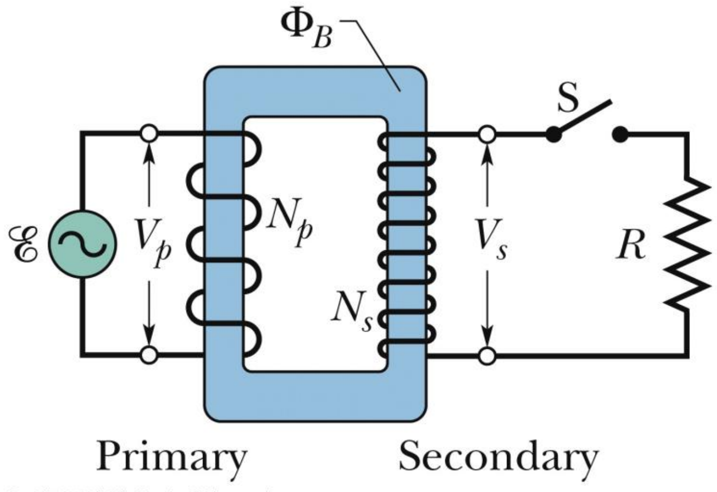
in which the term $\cos\phi$ is called the **power factor.** Because $\cos\phi = \cos(-\phi)$, Eq. 31-76 is independent of the sign of the phase constant ϕ.

To maximize the rate at which energy is supplied to a resistive load in an *RLC* circuit, we should keep the power factor $\cos\phi$ as close to unity as possible.

31-6 Transformers

A transformer (assumed to be ideal) is an iron core on which are wound a primary coil of N_p turns and a secondary coil of N_s turns. If the primary coil is connected across an alternating-current generator, the primary and secondary voltages are related by

$$V_s = V_p \frac{N_s}{N_p}$$



An ideal transformer (two coils wound on an iron core) in a basic trans- former circuit. An ac generator produces current in the coil at the left (the primary). The coil at the right (the secondary) is connected to the resistive load R when switch S is closed.

Energy Transfers. The rate at which the generator transfers energy to the primary is equal to I_pV_p. The rate at which the primary then transfers energy to the secondary (via the alternating magnetic field linking the two coils) is I_sV_s. Because we assume that no energy is lost along the way, conservation of energy requires that

$$I_pV_p = I_sV_s. \Rightarrow I_s = I_p \frac{N_p}{N_s} \Rightarrow I_p = \frac{1}{R}\left(\frac{N_s}{N_p}\right)^2 V_p.$$

The equivalent resistance of the secondary circuit, as seen by the generator, is

$$R_{eq} = \left(\frac{N_p}{N_s}\right)^2 R.$$

Sample Problem 31.08 Transformer: turns ratio, average power, rms currents

A transformer on a utility pole operates at $V_p = 8.5$ kV on the primary side and supplies electrical energy to a number of nearby houses at $V_s = 120$ V, both quantities being rms values. Assume an ideal step-down transformer, a purely resistive load, and a power factor of unity.

(a) What is the turns ratio N_p/N_s of the transformer?

KEY IDEA

The turns ratio N_p/N_s is related to the (given) rms primary and secondary voltages via Eq. 31-79 ($V_s = V_p N_s/N_p$).

Calculation: We can write Eq. 31-79 as

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}. \tag{31-83}$$

(Note that the right side of this equation is the *inverse* of the turns ratio.) Inverting both sides of Eq. 31-83 gives us

$$\frac{N_p}{N_s} = \frac{V_p}{V_s} = \frac{8.5 \times 10^3 \text{ V}}{120 \text{ V}} = 70.83 \approx 71. \qquad \text{(Answer)}$$

(b) The average rate of energy consumption (or dissipation) in the houses served by the transformer is 78 kW. What are the rms currents in the primary and secondary of the transformer?

KEY IDEA

For a purely resistive load, the power factor $\cos\phi$ is unity; thus, the average rate at which energy is supplied and dissipated is given by Eq. 31-77 ($P_{avg} = \mathscr{E}I = IV$).

Calculations: In the primary circuit, with $V_p = 8.5$ kV, Eq. 31-77 yields

$$I_p = \frac{P_{avg}}{V_p} = \frac{78 \times 10^3 \text{ W}}{8.5 \times 10^3 \text{ V}} = 9.176 \text{ A} \approx 9.2 \text{ A}. \qquad \text{(Answer)}$$

Similarly, in the secondary circuit,

$$I_s = \frac{P_{avg}}{V_s} = \frac{78 \times 10^3 \text{ W}}{120 \text{ V}} = 650 \text{ A}. \qquad \text{(Answer)}$$

You can check that $I_s = I_p(N_p/N_s)$ as required by Eq. 31-80.

(c) What is the resistive load R_s in the secondary circuit? What is the corresponding resistive load R_p in the primary circuit?

One way: We can use $V = IR$ to relate the resistive load to the rms voltage and current. For the secondary circuit, we find

$$R_s = \frac{V_s}{I_s} = \frac{120 \text{ V}}{650 \text{ A}} = 0.1846\ \Omega \approx 0.18\ \Omega. \qquad \text{(Answer)}$$

Similarly, for the primary circuit we find

$$R_p = \frac{V_p}{I_p} = \frac{8.5 \times 10^3 \text{ V}}{9.176 \text{ A}} = 926\ \Omega \approx 930\ \Omega. \qquad \text{(Answer)}$$

Second way: We use the fact that R_p equals the equivalent resistive load "seen" from the primary side of the transformer, which is a resistance modified by the turns ratio and given by Eq. 31-82 ($R_{eq} = (N_p/N_s)^2 R$). If we substitute R_p for R_{eq} and R_s for R, that equation yields

$$R_p = \left(\frac{N_p}{N_s}\right)^2 R_s = (70.83)^2(0.1846\ \Omega)$$
$$= 926\ \Omega \approx 930\ \Omega. \qquad \text{(Answer)}$$

31 Summary

LC Energy Transfer

- In an oscillating LC circuit, instantaneous values of the two forms of energy are

$$U_E = \frac{q^2}{2C} \quad \text{and} \quad U_B = \frac{Li^2}{2}$$ **Eq. 31-1&2**

LC Charge and Current Oscillations

- The principle of conservation of energy leads to

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0$$ **Eq. 31-11**

- The solution of Eq. 31-11 is

$$q = Q\cos(\omega t + \phi)$$ **Eq. 31-12**

- the angular frequency v of the oscillations is

$$\omega = \frac{1}{\sqrt{LC}}.$$ **Eq. 31-4**

Damped Oscillations

- Oscillations in an LC circuit are damped when a dissipative element R is also present in the circuit. Then

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$ **Eq. 31-24**

- The solution of this differential equation is

$$q = Qe^{-Rt/2L}\cos(\omega' t + \phi),$$ **Eq. 31-25**

Alternating Currents; Forced Oscillations

- A series RLC circuit may be set into forced oscillation at a driving angular frequency by an external alternating emf

$$\mathscr{E} = \mathscr{E}_m \sin \omega_d t.$$ **Eq. 31-28**

- The current driven in the circuit is

$$i = I\sin(\omega_d t - \phi)$$ **Eq. 31-29**

31 Summary

Series RLC Circuits

- For a series RLC circuit with an alternating external emf and a resulting alternating current,

$$I = \frac{\mathscr{E}_m}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathscr{E}_m}{\sqrt{R^2 + (\omega_d L - 1/\omega_d C)^2}}$$

Eq. 31-60&63

- and the phase constant is,

$$\tan\phi = \frac{X_L - X_C}{R}$$

Eq. 31-65

- The impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

Eq. 31-61

Power

- In a series RLC circuit, the average power of the generator is,

$$P_{avg} = I_{rms}^2 R = \mathscr{E}_{rms} I_{rms} \cos\phi.$$

Eq. 31-71&76

Transformers

- Primary and secondary voltage in a transformer is related by

$$V_s = V_p \frac{N_s}{N_p}$$

Eq. 31-79

- The currents through the coils,

$$I_s = I_p \frac{N_p}{N_s}$$

Eq. 31-80

- The equivalent resistance of the secondary circuit, as seen by the generator, is

$$R_{eq} = \left(\frac{N_p}{N_s}\right)^2 R,$$

Eq. 31-82