1. DATA AND ERROR ANALYSIS

OBJECT:

To review various aspects such as the meaning of the word "error", absolute and relative uncertainty, the propagation of errors, mean value, standard deviation, and proper graph, linear and logarithmic scales.

APPARATUS:

No measurement to be done. The various measurements are given in the text. Students are to analyse these results in order to exercise the various aspects of error analysis.

THEORY:

There is always some uncertainty in a measurement basically for two reasons:

(1) Statistical fluctuations in measured quantity, and
(2) Inaccuracies in our measurements.

In this text the word "error" refers to such uncertainty. For example, the height of a water column coming out of a water hose fluctuates with time due to changes in the water pressure in the hose. Measurement of the height, no matter how accurate, will have a statistical distribution about a mean value. On the other hand, if we want to measure the length of a fish swimming in the water, measurements will be inaccurate due to the motion of the fish, even though the length of the fish remains unchanged.

Here, let us name the quantity to be measured x and its mean value x_o. Absolute uncertainty Δx and relative uncertainty $\Delta x/x_o$ are commonly used as a measure of the accuracy of our measurements.

Absolute uncertainty, Δx, in this text will have a loose meaning, in that it will sometimes be termed "standard deviation", "absolute error", "root mean square", "r.m.s", etc. The value of Δx is assigned according to the circumstances. For example, if the statistical distribution of x_i $(i=1,2,...n)$ is given, then x_o and Δx are defined as:

$$x_o = \frac{\sum_{i=1}^{n} x_i}{n} \quad \text{(mean value, or average value)}$$

and

$$\Delta x = \sqrt{\frac{\sum_{i=1}^{n} (x_i - x_o)^2}{n - 1}} \quad \text{(standard deviation)}$$

However, in some cases, there are not enough data to find x_o and Δx from the above formulas. We may only have a single reading: for example, measuring the time it takes for the sound of lightning to reach the observer after the lightning strikes. Since no similar event exists, the uncertainties are assigned on the basis of the measuring devices and methods employed. If we used an ordinary wrist watch the accuracy could not be better than a second, therefore, $\Delta t \approx 1$ sec.

PROPAGATION OF UNCERTAINTIES:

If the result we seek, say z, is a function of several measurable quantities, such as x,y,t, etc., then we adopt with caution, the following rule to find the absolute uncertainty, Δz, in terms of Δx, Δy, Δt, etc.:

$$\Delta z = \left|\frac{\partial z}{\partial x}\right|_0 \Delta x + \left|\frac{\partial z}{\partial y}\right|_0 \Delta y + \left|\frac{\partial z}{\partial t}\right|_0 \Delta t + \ldots$$

where the absolute values of the partial derivatives are evaluated at the mean values x_0, y_0, t_0, etc., and Δx, Δy, Δt, etc. are the absolute uncertainties for the corresponding measurable variables.

Example: Centripetal force F is given by:

$$F = \frac{mv^2}{r}$$

Here we assume m, v, and r are measurable variables. Then, ΔF and $\Delta F/F_0$ are found as follows:

$$\Delta F = \left|\frac{\partial F}{\partial m}\right|_0 \Delta m + \left|\frac{\partial F}{\partial v}\right|_0 \Delta v + \left|\frac{\partial F}{\partial r}\right|_0 \Delta r$$

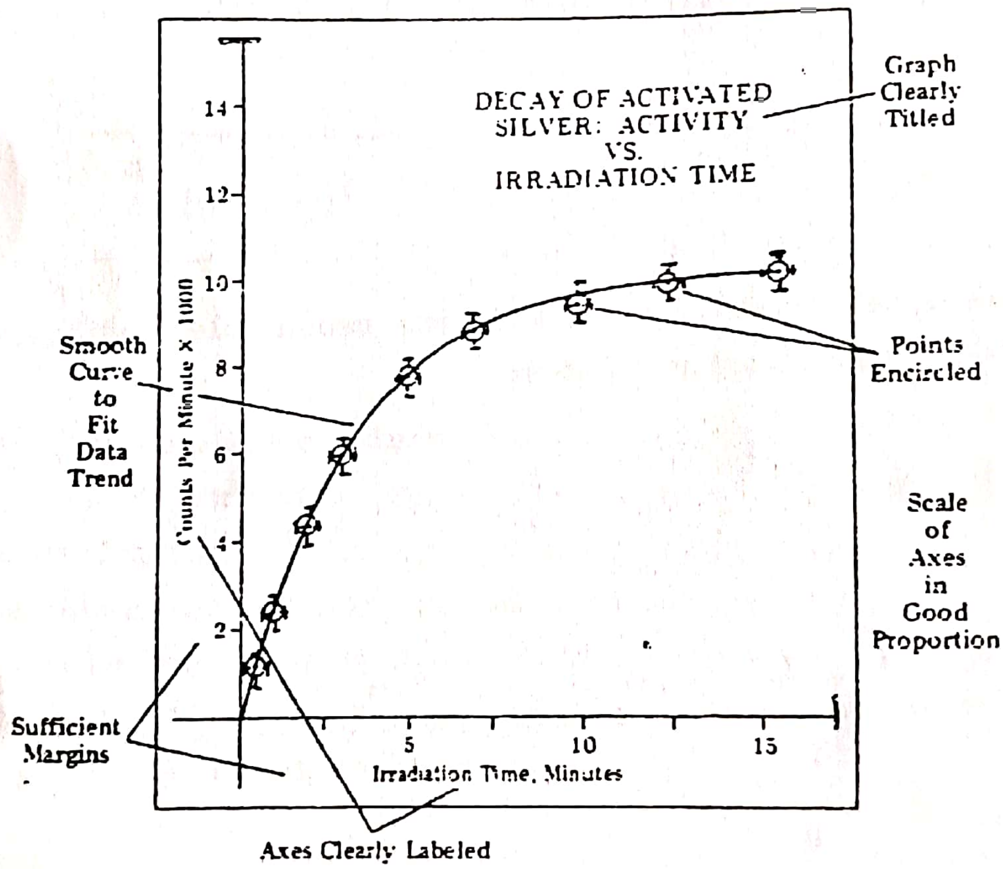
$$\Delta F = \frac{v_0^2}{r_0} \Delta m + \frac{2m_0 v_0}{r_0} \Delta v + \frac{m_0 v_0^2}{r_0^2} \Delta r, \quad \text{and}$$

$$\frac{\Delta F}{F_0} = \frac{\Delta F}{\left(\frac{m_0 v_0^2}{r_0}\right)} = \frac{\Delta m}{m_0} + 2\frac{\Delta v}{v_0} + \frac{\Delta r}{r_0}$$

Here m_o, v_o, r_o, and F_o are the mean values, and Δm, Δv, Δr and ΔF are the absolute uncertainties. Also notice that the last term is taken to be positive because of the absolute partial derivatives.

THE PROPER GRAPH:

When plotting a graph, the following rules are recommended:



A Proper Graph.

The figure given above is an example of a graph plotted on a linear scale. In some instances, instead of numbers their logarithms are used on a linear scale, or, equivalently, the numbers themselves are used on a logarithmic scale. Exercise 4 demonstrates this point nicely. There are basically two reasons for using a logarithmic scale:

(1) If the numbers used span too large of a region, such as from 1 to 10^6, then by using their logarithms (or by using a logarithmic scale) the range is transferred from 0 to 6. This allows us to see the small as well as the large variations in the numbers on a single scale.

(2) It allows us to obtain a linear graph between y and x connected to each other through

$$y = \beta x^{\alpha}.$$

Since $\log y = \log \beta + \alpha \log x$, log y vs log x will yield a linear graph, in that the origin corresponds to $y = x = 1$ ($\log 1=0$), $\log \beta$ is the intersection, and α is the slope of the line. Refer to exercise 4 for more detail. In the following, five exercises are given to practice the various aspects of error analysis described above.

Exercise 1

In a calorimetry experiment to measure the latent heat of fusion of water, L, a known mass of ice, m_i, is added to a known mass of water m_w, contained in an insulated beaker. The initial and final temperatures of the water are recorded and L is computed from the equation

$m_i L + m_i T_2 = m_w (T_1 - T_2)$ calories, where

m_i = mass of ice = 14.2 ± 0.1 g

m_w = mass of water = 72.3 ± 0.1 g

T_1 = initial temperature of water = $(25.4 \pm 0.1)^{\circ}$C

T_2 = final temperature of water = $(7.8 \pm 0.1)^{\circ}$C

Calculate L (in cal/g), the absolute uncertainty in L, and the percentage uncertainty in L.

Exercise 2

The index of refraction, n, of a material can be calculated by measuring the angle of minimum deviation, D, of a parallel beam of light incident on a prism of apex angle, A, made of the material, and making use of the formula

$$n = \frac{\sin\left(\frac{A+D}{2}\right)}{\sin\left(\frac{A}{2}\right)}$$

If the angle A is measured to be $60^{0} \pm 2'$, and the angle D is measured to be $23^{0}14' \pm 2'$, calculate values of n and δn.

Exercise 3

The fundamental frequency of oscillation, f, of a sonometer wire of length ℓ under a tensile force T is given by

$$f = \frac{1}{2\ell}\sqrt{\frac{T}{\mu}}\ \text{Hz.}$$

where μ is the mass per unit length of the wire. The following table gives the results of an experiment in which the tension in the wire was changed and the length of the wire was varied until, for each value of T, the sonometer vibrated at a fundamental frequency f = 100 Hz (exactly).

($T\pm0.01$) Nt	2.00	4.00	6.00	8.00	10.00	12.00	14.00
($\ell\pm0.05$) m	0.39	0.52	0.64	0.73	0.85	0.93	1.00

Plot an appropriate graph of the data to indicate a linear relationship. From the graph obtain a maximum, a minimum, and a best slope, and calculate values for μ and $\delta\mu$.

Exercise 4

Certain vacuum tube diodes exhibit current-voltage characteristics which may be represented by the equation

$$I = A\,V^n \text{ where}$$

I = current in amperes

V = voltage in volts

A and n = constants.

In order to determine the values of the constants A and n for a particular diode, the current through the diode was measured for various voltages across the diode. The following table lists the results.

V(volts) ± 10%	5	10	15	20	30	40	60
I (milliamp.)	6±1	16±2	33±3	47±4	85±5	148±10	270±20

Plot an appropriate graph of the data to indicate a linear relationship. From the graph obtain values for n, A, and δn using the techniques employed in exercise 3. Comment on the difficulties in determining δA.

Exercise 5

When measuring the radius 'a' of the capillary of a viscosimeter, the following measurements are obtained:

a = 0.085; 0.087; 0.085; 0.086; 0.085; 0.087; 0.086; 0.085; 0.086; 0.085 cm

Calculate the average $\bar{a}$ and the r.m.s. error e.