4 points

Q23) [Read the Book, solved there]

The wavefunction

$$\psi(x) = Cxe^{-\alpha x^2}$$

describes a state of the quantum oscillator, where the potential is given by

$$U(x) = \frac{1}{2}m\omega^2x^2.$$

(a) Using Schrödinger's equation, obtain an expression for α in terms of the oscillator mass m and the classical frequency of vibration ω.
(b) What is the energy of this state?
(c) Normalize this wave.
(d) Find the uncertainty in the position Δx. (+2)

(a) $\psi = Cxe^{-\alpha x^2}$

2

$$\frac{d\psi}{dx} = Ce^{-\alpha x^2} - 2\alpha Cx^2e^{-\alpha x^2}$$

$$\frac{d^2\psi}{dx^2} = -2\alpha Cxe^{-\alpha x^2} - 4\alpha Cxe^{-\alpha x^2} + 4\alpha^2Cx^3e^{-\alpha x^2}$$

$$= -6\alpha Cxe^{-\alpha x^2} + 4\alpha^2Cx^3e^{-\alpha x^2}$$

Shrödinger Equation $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi = E\psi$

$$\rightarrow -\frac{\hbar^2}{2m}(-6\alpha + 4\alpha^2x^2)\cancel{\psi} + \frac{1}{2}m\omega^2x^2\cancel{\psi} = E\cancel{\psi}$$

$$\Rightarrow \left(-\frac{4\hbar^2\alpha^2}{2m} + \frac{1}{2}m\omega^2\right)x^2 + \left(\frac{6\alpha\hbar^2}{2m} - E\right) = 0$$

$$\therefore \frac{1}{\cancel{2}}m\omega^2 = \frac{4\hbar^2\alpha^2}{\cancel{2}m} \Rightarrow \alpha^2 = \frac{m^2\omega^2}{4\hbar^2} \quad \text{or} \quad \alpha = \frac{m\omega}{2\hbar}$$

(b) $E = \frac{6\hbar^2}{2m}\frac{m\omega}{2\hbar} = \frac{3}{2}\hbar\omega$

1

(c) $$\int_{-\infty}^{\infty}\psi^2dx = 1 = C^2\int_{-\infty}^{\infty}x^2e^{-2\alpha x^2}dx = +2C^2\underbrace{\int_0^{\infty}x^2e^{-2\alpha x^2}dx}_{\text{formula sheet}}$$

$$1 = 2C^2\frac{1}{8\alpha}\sqrt{\frac{\pi}{2\alpha}} \Rightarrow C = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}$$

1

(d) $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi \, dx = C^2 \int_{-\infty}^{\infty} x^3 e^{-2\alpha x^2} dx$$

$$= C^2 \left\{ \int_{-\infty}^{0} x^3 e^{-2\alpha x^2} dx + \int_{0}^{\infty} x^3 e^{-2\alpha x^2} dx \right\}$$

$x \to -y$

$$= C^2 \left\{ - \int_{0}^{\infty} y^3 e^{-2\alpha y^2} dy + \int_{0}^{\infty} x^3 e^{-2\alpha x^2} dx \right\} = 0$$

odd function

+2 points

$$\langle x^2 \rangle = C^2 \int_{-\infty}^{\infty} x^4 e^{-2\alpha x^2} dx = 2C^2 \int_{0}^{\infty} \frac{y^4}{4\alpha^2} e^{-y^2} \frac{dy}{\sqrt{2\alpha}} \quad ; y = \sqrt{2\alpha}\, x$$

$$= \frac{2C^2}{(2\alpha)^{5/2}} \int_{0}^{\infty} y^4 e^{-y^2} dy = \frac{2C^2}{(2\alpha)^{5/2}} \frac{3\sqrt{\pi}}{8}$$

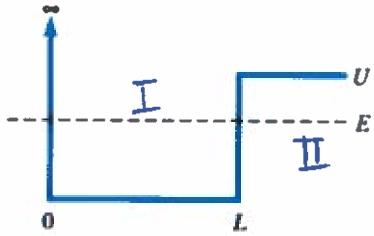
$$\Delta x = \sqrt{\langle x^2 \rangle}$$

(3 points)

Q24) [proved for all cases] Consider a square well having an infinite wall at $x = 0$ and a wall of height U at $x > L$. For the case $E < U$, obtain solutions to the Schrödinger equation inside the well $(0 < x < L)$ and in the region beyond $(x > L)$ that satisfy the appropriate boundary conditions at $x = 0$ and $x = \infty$. Find a relation that can be used to obtain the allowed energy levels of this system.

region I $\quad 0 \leq x \leq L, \; U = 0$

region II $\quad x > L \quad , \; U = U_0$



for region I $\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$

or $\quad \frac{d^2\psi}{dx^2} + \underbrace{\frac{2mE}{\hbar^2}}_{k^2}\psi = 0 \quad = \frac{d^2\psi}{dx^2} + k^2\psi = 0$

1

$\Rightarrow \quad \psi_I(x) = A\sin kx + B\cos kx$

at $x = 0 \quad \psi_I(x=0) = 0 = B \rightarrow \boxed{\psi_I(x) = A\sin kx}$

for region II $\Rightarrow \quad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi$

or $\quad \frac{d^2\psi}{dx^2} - \underbrace{\frac{2m}{\hbar^2}(U-E)}_{\alpha^2}\psi = 0 \quad = \frac{d^2\psi}{dx^2} - \alpha^2\psi = 0$

$\Rightarrow \quad \psi_{II}(x) = Ce^{-\alpha x} + De^{\alpha x}$

1

at $x \rightarrow \infty \quad \psi_{II}(\infty) = 0 = De^{\infty} \Rightarrow D = 0$

$\therefore \quad \psi_{II}(x) = Ce^{-\alpha x}$

1

at $x = L$:

$\rightarrow \psi_I(L) = \psi_{II}(L) \Rightarrow A\sin kL = Ce^{-\alpha L}$

$\rightarrow \psi_I'(L) = \psi_{II}'(L) \Rightarrow Ak\cos kL = -C\alpha e^{-\alpha L}$

Divide

~~kcotkL~~

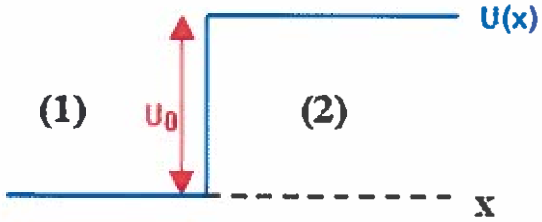
$\boxed{k\cot kL = -\alpha}$

3 points

Q) **[Solved in the Lecture]** For particles incident on a step potential with $E < U_0$, show that transmission probability for the particle is $T = 0$. Justify your answer.

region I, $U=0 \quad x<0$

region II, $U=U_0 \quad x>0$



for region I:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I}{dx^2} = E\psi_I \Rightarrow \frac{d^2\psi_I}{dx^2} + k^2\psi_I = 0 \quad ; \ k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\therefore \psi_I = Ae^{ikx} + Be^{-ikx}$$

1

for region II

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{II}}{dx^2} + U\psi_{II} = E\psi_{II} \Rightarrow \frac{d^2\psi_{II}}{dx^2} - \alpha^2\psi_{II} = 0 \ ; \ \alpha = \frac{\sqrt{2m(U-E)}}{\hbar}$$

$$\psi_{II} = Ce^{-\alpha x}$$

Apply B.Cs at $x=0 \Rightarrow \quad A + B = C$, where $\psi_I(0) = \psi_{II}(0)$

& $ik(A-B) = -\alpha C$, where $\psi_I'(0) = \psi_{II}'(0)$

1

$$\Rightarrow -\alpha(A+B) = ik(A-B) \quad \text{or} \quad A(ik+\alpha) = B(ik-\alpha)$$

$$\Rightarrow \frac{B}{A} = \frac{ik+\alpha}{ik-\alpha} \quad \& \quad R = \left|\frac{B}{A}\right|^2 = \frac{k^2+\alpha^2}{k^2+\alpha} = 1$$

since $T + R = 1$ & $R = 1 \Rightarrow T = 0$ 1

3 points

Q) [suggested Problem Q21] Suppose that a hydrogen atom is in the $3s$ state. Taking $r = a_0$, calculate values for (a) $\psi_{3s}(a_o)$, (b)$|\psi_{3s}(a_o)|^2$, and (c) $P_{3s}(a_0)$.

$$\psi_{3s} = \frac{1}{\sqrt{4\pi}}\frac{2}{(3a_0)^{\frac{3}{2}}}\left(1 - \frac{2r}{3a_0} - \frac{2r^2}{27a_0^2}\right)e^{-\frac{r}{3a_0}}$$

(a) $\psi_{3s}(a_0) = \frac{1}{\sqrt{4\pi}} \frac{2}{(3a_0)^{3/2}} \left(1 - \frac{2}{3} - \frac{2}{27}\right) e^{-1/3}$ (1)

$= \frac{1}{\sqrt{\pi}} \left(\frac{1}{3a_0}\right)^{3/2} \frac{7}{27} e^{-1/3} = 5.24 \times 10^{13}\ m^{-3/2}$

(b) $|\psi_{3s}(a_0)|^2 = (5.24 \times 10^{13})^2 = 2.75 \times 10^{27}\ m^{-3}$ (1)

(c) $P_{3s}(a_0) = 4\pi a_0^2 |\psi_{3s}(a_0)|^2 = 4\pi a_0^2 \times 2.75 \times 10^{27}$

$= 9.66 \times 10^{7}\ m^{-1}$. (1)

3 points

Q22) [Example in the lecture] The radial part of the wavefunction for the hydrogen atom in the $3d$ state is given by

$$R_{3d}(r) = Cr^2 e^{-r/3a_0},$$

where C is a constant and a_0 is the Bohr radius. Using this expression, calculate the average value of r for an electron in this state.

$$P(r) = r^2 |R_{3d}|^2 = C^2 r^6 e^{-2r/3a_0} \quad 1$$

$$\langle r \rangle = \int_0^\infty r\,P(r)\,dr = C^2 \int_0^\infty r^7 e^{-2r/3a_0}\,dr$$

1

$$\text{let } \frac{2r}{3a_0} = z \Rightarrow r = \frac{3a_0}{2} z \;\;\&\;\; dr = \frac{3a_0}{2}\,dz$$

$$\Rightarrow \langle r \rangle = C^2 \left(\frac{3a_0}{2}\right)^8 \int_0^\infty z^7 e^{-z}\,dz = C^2 \left(\frac{3a_0}{2}\right)^8 7!$$

1

$$= 35\,\frac{3^{10}}{2^4}\,a_0^8 C^2 \;\; *$$

3 points

Q) **[simple and gift]** What are the possible values of m_ℓ if (a) $\ell = 3$, and (b) $\ell = 4$, and (c) $\ell = 0$? Compute the minimum possible energy for each case for the Li^{2+} ion.

(a) $l = 3 \Rightarrow m_l = 0, \pm 1, \pm 2, \pm 3$

(b) $l = 4 \Rightarrow m_l = 0, \pm 1, \pm 2, \pm 4$ ①

(c) $l = 0 \Rightarrow m_l = 0$

for $n \Rightarrow l = 0, 1, 2, ---, n-1$

(a) $l = 3 \rightarrow n = 4$

$$E_4 = -\frac{13.6 Z^2}{n^2} = -\frac{13.6 \times 9}{16} = -7.65 \text{ eV}$$

(b) $l = 4 \Rightarrow n = 5$

$$E_5 = -\frac{13.6 \times 9}{25} = -4.90 \text{ eV}$$ ②

(c) $l = 0 \Rightarrow n = 1$

$$E_1 = -13.6 \times 9 = -122.4 \text{ eV}$$

3 points

Q) [Example in Ch7 lecture] In a particular semiconductor device, electrons that are accelerated through a potential of 5.0 V attempt to tunnel through a barrier of width 0.80 nm and height 6.0 V. What fraction of the electrons are able to tunnel through the barrier if the potential is zero outside the barrier? Which is more effective in preventing tunneling, the barrier potential height or the barrier width?

$$T(E) = \left\{ 1 + \frac{1}{4}\left(\frac{U^2}{E(U-E)}\right)\sinh^2(\alpha L) \right\}^{-1}$$

1

$$U = 6\,eV \;,\; E = 5\,eV \;,\; \alpha = \frac{\sqrt{2m(U-E)}}{\hbar} = 5.14 \times 10^9\ m^{-1}$$

$$\sinh(\alpha L) = \sinh(5.14 \times 10^9 * 0.8 \times 10^{-9}) = 30.47$$

$$T(E) = \left\{ 1 + \frac{1}{4}\,\frac{36}{5} * 30.47^2 \right\}^{-1} \sim 0.06\,\%$$

1

Increasing the width is more effective, since the particle will be in a forbidden region where it may decay before it has the probability to penetrate through the barrier.

1