$$\psi(x,t) = \psi(x)\,\phi(t) = \psi(x)\, e^{-i\omega t}$$

$$\frac{-\hbar}{2m} \frac{d^2 \psi(x)}{d x^2} + U(x)\,\psi(x) = E\,\psi(x)$$

① Free Particle.

$$U(x) = 0$$

$$\frac{-\hbar}{2m} \frac{d^2 \psi}{d x^2} = E\,\psi$$

k^2

$$\frac{d^2 \psi}{dx} + \frac{2Em}{\hbar^2}\,\psi = 0$$

Solution:

$$\psi(x) = A e^{ikx}$$

Energy:

$$E = \frac{\hbar^2 k^2}{2m}$$

not qaunized!

(2) the Particle in a Box 1-D

$$U \begin{cases} 0, & 0<x<L \\ \infty, & \text{otherwise} \end{cases}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi$$

for $0<x<L$

$U=0$

$$\frac{d^2\psi}{dx^2} + \underbrace{\frac{2m}{\hbar^2}E}_{=k^2}\psi = 0$$

Solution:

$$\psi = A\sin kx + B\cos kx$$

B.C.

$\psi = 0 \Longrightarrow x=0,\ x=L$

$B=0$

$0 = A \sin KL$

$KL = n\pi$

$$\boxed{K = \frac{n\pi}{L}}$$

the constant A

$$\int_0^L \psi^2 = 1$$

$$A^2 \int_0^L \sin^2 \frac{n\pi}{L} x = 1$$

$$\downarrow$$

$$\boxed{A = \sqrt{\frac{2}{L}}}$$

Energy:

$$K = \frac{n\pi}{L}$$

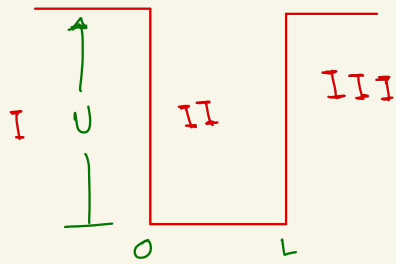
$$K^2 = \frac{2mE}{\hbar^2}$$

$$\frac{n^2\pi^2}{L^2} = \frac{2mE}{\hbar^2}$$

$$\boxed{E = \frac{n^2\pi^2\hbar^2}{2mL^2}}$$

③ The finite square well

$E < U$



$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi$$

$\boxed{II}$ $U = 0, \quad 0 < x < L$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} + \underbrace{\frac{2Em}{\hbar^2}}_{=k^2}\psi = 0$$

Solution:

$$\psi = C\sin kx + D\cos kx$$

I, III

$E < U$

$U = U.\ x > L,\quad x < 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - U)\psi = 0$$

$$\Downarrow$$

$$\frac{d^2\psi}{dx^2} - \underbrace{\frac{2m}{\hbar^2}(U - E)}_{= \alpha^2}\psi = 0$$

$$\frac{d^2\psi}{dx^2} - \alpha^2\psi = 0$$

Solution: $\psi_I = A e^{\alpha x} \longrightarrow x < 0$

$\psi_{III} = B e^{-\alpha x} \longrightarrow x > 0$

To Decay

Constants A, B, C, D

$\psi_{I}(x=0) = \psi_{II}(x=0)$ ①

$\psi_{II}(x=L) = \psi_{III}(x=L)$ ②

$\left.\frac{d\psi_I}{dx}\right|_{x=0} = \left.\frac{d\psi_{II}}{dx}\right|_{x=0}$ ③

$\left.\frac{d\psi_I}{dx}\right|_{x=L} = \left.\frac{d\psi_{III}}{dx}\right|_{x=L}$ ④

the energies

$$\boxed{C \sin kL + D \cos kL = B e^{-\alpha L}}$$

$\div$

$$\boxed{C k \cos kL - D k \sin kL = -\alpha B e^{-\alpha L}}$$

$$k \frac{C \cos kL - D \sin kL}{C \sin kL + D \cos kL} = -\alpha$$

does not exist, then

$\alpha = \sqrt{\frac{2m}{\hbar^2}(U-E)}$

$k = \sqrt{\frac{2mE}{\hbar^2}}$

$$\boxed{E_n = \frac{n^2 \pi^2 \hbar^2}{2m(L+2\delta)^2}}$$

$$\delta = \frac{1}{\alpha}$$

4 The quantum oscillator

$V(x) = \frac{1}{2} k x^2$

in C.M. the particle like being at the edges

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U\psi = E\psi$$

in Q.M. the particle like to stay at the equilibrum

$U(x) = \frac{1}{2} m w^2 x^2$

$$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2} U\psi + \frac{2m}{\hbar^2} E\psi = 0$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\psi(E-U) = 0$$

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}\psi(U-E)$$

$U \rightarrow \frac{1}{2} m w^2 x^2$

Solution:

$$\psi(x) = C_0 e^{-\alpha x^2}$$

$\psi(x) = 0$ if $\pm\infty = x$

$\psi(x) =$ finite if $x = 0$

$$\psi' = -2C_0 e^{-\alpha x^2} \cdot \alpha x$$

$$\psi' = -2\alpha C_0 \, x e^{-\alpha x^2}$$

$$\psi'' = -2\alpha C_0 \left[e^{-\alpha x^2} - 2x^2 \alpha e^{-\alpha x^2} \right]$$

$$\psi'' = -2\alpha C_0 e^{-\alpha x^2} + 4x^2 \alpha^2 C_0 e^{-\alpha x^2}$$

$$\psi'' = \underbrace{C_0 e^{-\alpha x^2}}_{=\psi} \left[4x^2\alpha^2 - 2\alpha \right]$$

$$\psi'' = \cancel{\psi}\left(4x^2\alpha^2 - 2\alpha \right) = \frac{2m}{\hbar^2}\left(U - E \right) \cancel{\psi}$$

$$4x^2\alpha^2 = \frac{2m}{\hbar^2} U$$

$$4x^2\alpha^2 = \frac{2m}{\hbar^2} \frac{1}{2} m w^2 x^2$$

$$\alpha^2 = \frac{m^2 w^2}{\hbar^2 \, 4} = \alpha = \frac{m w}{2\hbar}$$

$$2\alpha = \frac{2m}{\hbar^2} E$$

$$\frac{m\omega}{\hbar} = \frac{2m}{\hbar^2} E$$

$$E = \frac{1}{2}\hbar\omega$$

normalization to find C_0

$$\int_{-\infty}^{\infty} \psi^2 = 1$$

$$C_0^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}\,dx = 1$$

$$\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

$$C_0^2 \sqrt{\frac{\hbar\pi}{m\omega}} = 1$$

$$C_0^2 = \sqrt{\frac{m\omega}{\hbar\pi}}$$

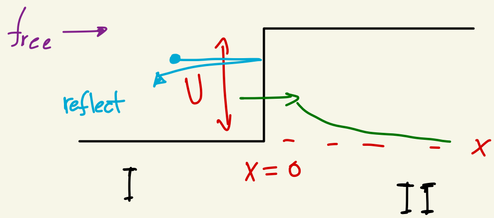
$$C_0 = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$$

$$\psi = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

for the ground state

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

5 Potential steP



$E < U \qquad x < 0, \; U = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\,\psi(x)$$

$$\frac{d^2\psi}{dx^2} + \overbrace{\frac{2mE}{\hbar^2}}^{k^2}\psi = 0$$

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

$$\psi(x) = \underbrace{A e^{ikx}}_{\substack{\text{Right} \\ \text{incident}}} + \underbrace{B e^{-ikx}}_{\substack{\text{Left} \\ \text{Reflected}}}$$

$E < U \qquad x > 0, \; U = U$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx} + U\psi = E\psi$$

$$\frac{d^2\psi}{dx^2} - \frac{2m}{\hbar^2}(U - E)\psi = 0$$

$$\frac{d^2\psi}{dx^2} - \alpha^2\psi = 0$$

$$\alpha = \frac{2m}{\hbar^2}(U - E)$$

$$\psi(x) = C e^{-\alpha x} + \cancel{D e^{\alpha x}} \; 0$$

because the Particle should decay at ∞

$E > U$

$x < 0, \quad U = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\psi'' + \frac{2m}{\hbar^2}E\psi = 0$$

$$\psi_I = Ae^{ikx} + Be^{-ikx}$$

$E > U$

$x > 0, \quad U = U$

$$-\frac{\hbar^2}{2m}\psi'' + U\psi = E\psi$$

$$\psi'' - \frac{2m}{\hbar^2}U\psi + \frac{2m}{\hbar^2}E\psi = 0$$

$$\psi'' + \underbrace{\frac{2m}{\hbar^2}(E-U)}_{k'^2}\psi = 0$$

$$\psi = Ce^{ik'x} + \cancel{D}e^{-ik'x}$$

$D = 0$

because it will not Reflect