KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DEPARTMENT OF PHYSICS

PHYS.300 – MAJOR EXAM -II (TERM 211)

Instructor: Dr. Hocine Bahlouli

Student Name: ____________________________

ID. No. : ____________________

- **Exam time : up to a maximum of 90 Minutes**
- **Solve the following three problems and show all details and intermediate steps to gain full credit.**

Problem #	Grade
1	/33
2	/34
3	/33
Total	**/100**

Q.1. Consider the following general form of **functional**

$$J = \int_{i}^{f} f(y)ds = \int_{x_i}^{x_f} f(y)\sqrt{1+y'(x)^2}\,dx\,;\; ds = \sqrt{dx^2+dy^2}\,;\; y(x_i) = y_i\,;\; y(x_f) = y_f$$

which we would like to **minimize** with **fixed end points** y_i and y_f, ds is an element arc length in xy-plane and ***f(y)*** is a **given function** of the curve or path ***y(x)***.

a) **Show** that the function *y(x)* that **extremizes** the functional J **satisfies** the following differential equation

$$1+[y'(x)]^2 = Af^2(y) \quad ; \quad y'(x) = \frac{dy}{dx} \qquad \textbf{(12pts)}$$

where A is an integration constant to be determined by the boundary conditions on *y(x)*.
Show the above result using **both Euler equation and it second form.**

b) **Integrate** the above differential equation and express *x(y)* in its integral form. **(5pts)**

c) **Solve** the above differential equation for *y(x)* in the particular case when $f(y)=1$, give an **interpretation** for this result and give **one example** of a **physical phenomenon** that falls in this category. **(8pts)**

d) **Find** the path *y(x)* followed by a particle that minimizes J when $f(y) = \sqrt{y}$. **(8pts)**

Q. 2 Consider **Gauss theorem** for a mass density ρ, Gauss surface of surface S and volume V

$$\oint_S \vec{g}\cdot d\vec{a} = -4\pi G m_{enclosed} = -4\pi G\int_V \rho dV \qquad (1)$$

a) **Explain** how symmetry considerations ensure that $\vec{g}(\vec{r}) = -g(r)\hat{r}$, at a distance **r** from an **infinite vertical** wire of uniform **linear** mass density λ_1, $\hat{r}$ being the radial unit vector away from the wire. **(5 pts)**

b) Let us **apply Gauss theorem** to compute the gravitational field $\vec{g}$ at a distance **r** from an **infinite vertical** wire of uniform **linear** mass density λ_1. **(7 pts)**

c) Consider a **second horizontal** wire of length **L** and uniform linear mass density λ_2 at a distance **D** from the vertical wire. Find the **net gravitational force** on the horizontal wire from the infinite one. **(10 pts)**

d) Study the above **gravitational force** in the limit **L<< D**, and **explain** the simplicity of this result knowing that the total mass of the horizontal wire is **M=** λ_2**L** (use $\ln(1+x) \sim x$ as $x \to 0$). **(5 pts)**

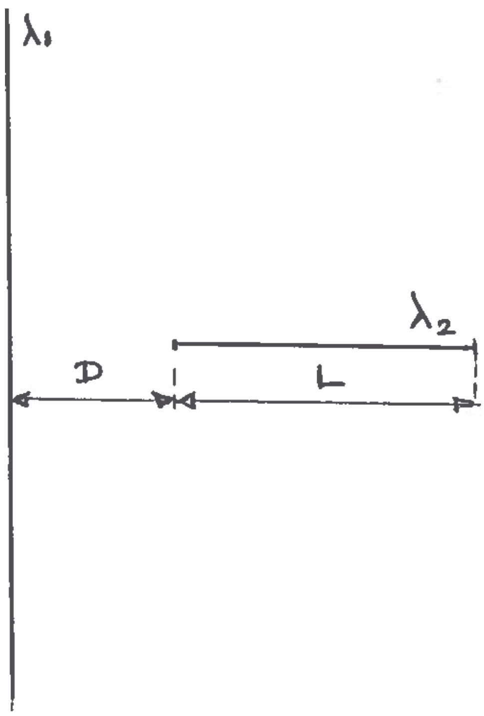
e) If you are given a **spherically symmetric** gravitational potential g(r), r being the radial distance from the center of a spherical mass distribution. Using (1) and the divergence theorem: **(7 pts)**

$$\oint_S \vec{A}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{A}\, dV \qquad (2)$$

show how you can obtain the radial mass density ρ(r) from (1).

Apply you results to find ρ(r) associated with a given $g(r) = \dfrac{b}{a^2 + r^2}$ where a and b are given positive constants.

Count this question as a bonus



Q. 3 Consider a **simple pendulum** in a vertical plane (mass m_2, length **b**) whose point of support is a mass m_1 which can slide **horizontally on a frictionless** surface and whose position **is given** by $x(t)$. Let us compute the **Lagrangian and Euler Lagrange equations.** Every part of our system is assumed to be massless except for the two point masses m_1 and m_2.

a) Explain why the system has only **two** degrees of freedom **(x, θ)** and **show** that its Lagrangian can be written as **(13 pts)**

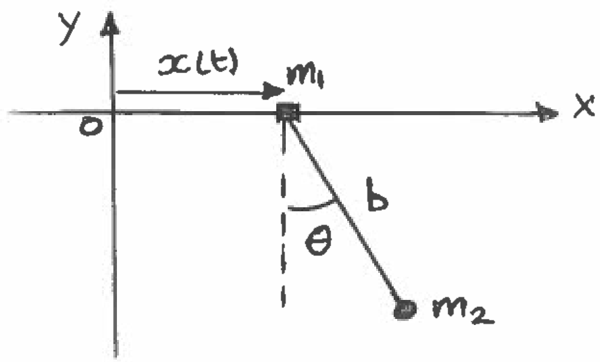
$$L(x,\theta,\dot{x},\dot{\theta};t)=\frac{1}{2}(m_1+m_2)\dot{x}^2+\frac{1}{2}m_2(2b\dot{x}\dot{\theta}\cos\theta+b^2\dot{\theta}^2)+m_2gb\cos\theta.$$

b) Write down **Lagrange equations** of motion for both x and θ and deduce that the quantity **(10 pts)**

$$M=(m_1+m_2)\dot{x}+m_2b\dot{\theta}\cos\theta$$

is conserved, what is its **physical meaning? Deduce** the equation of motion for θ using $\omega_0^2=g/b$.

c) Suppose that the point of support of the pendulum is forced to oscillate with a **given** amplitude A and frequency ω, that is $x(t)=A\cos\omega t$. Deduce the new **Lagrangian** in its most simple form, and **solve the equation** of motion for small $\theta(t)$ using results from chapter 3. **(10 pts)**



Major Exam II — Fall 211

Phys. 300

Q1. $J = \int_{x_i}^{x_f} f(y)\sqrt{1+y'(x)^2}\,dx = \int_{x_i}^{x_f} F(y,y';x)\,dx$

a/ $y(x)$ that extremize the function J satisfy Euler equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

Since x is a cyclic variable $\frac{\partial F}{\partial x} = 0$ then the second form of Euler equation is more suitable

$$y'\frac{\partial F}{\partial y'} - F = \text{Constant} = C$$

$$\Rightarrow y'\frac{f(y)\,y'}{\sqrt{1+y'^2}} - f(y)\sqrt{1+y'^2} = \frac{f(y)\,y'^2 - f(y)(1+y'^2)}{\sqrt{1+y'^2}} = -\frac{f(y)}{\sqrt{1+y'^2}}$$

thus $\frac{f(y)}{\sqrt{1+y'^2}} = C \Rightarrow \boxed{1+y'(x)^2 = A f^2(y)}$; $A = \text{Constant}$

$$\Rightarrow \frac{dy}{dx} = \pm\sqrt{A f^2(y) - 1}$$

$$\Rightarrow x = \pm\int \frac{dy}{\sqrt{A f^2(y) - 1}}$$

To obtain an explicit solution $x(y)$ we need an explicit form for $f(y)$ which will be treated in the following questions, then

What about using the first form of Euler equation,

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y}\sqrt{1+y'^2} \quad ; \quad \frac{\partial F}{\partial y'} = \frac{f(y)\,y'(x)}{\sqrt{1+y'^2(x)}} \quad ; \quad f'(y) = \frac{\partial f}{\partial y}$$

$$\frac{d}{dx}\frac{\partial F}{\partial y'} = \frac{f'(y)\,y'^2(x)}{\sqrt{1+y'^2(x)}} + \frac{f(y)\,y''(x)}{\sqrt{1+y'^2(x)}} - \frac{f(y)\,y'(x)^2\,y''(x)}{(1+y'^2(x))^{3/2}}$$

Euler Eq. $\Rightarrow f'(y)\sqrt{1+y'^2} - \frac{f'(y)\,y'^2(x)}{\sqrt{1+y'^2}} - \frac{f(y)\,y''(x)}{\sqrt{1+y'^2}} + \frac{f(y)\,y'^2(x)\,y''(x)}{(1+y'^2)^{3/2}} = 0$

multiply by $y'(x)$ to get the last eq. $\Rightarrow \frac{f'(y)}{(1+y'^2)^{1/2}} - \frac{f(y)\,y''(x)}{(1+y'^2)^{3/2}} = 0 \Rightarrow 0 = \frac{d}{dx}\left[f(y)(1+y'^2)^{-1/2}\right]$

$$\Rightarrow f(y)(1+y'^2)^{-1/2} = C \Rightarrow 1+y'^2(x) = A f^2(y)$$

Over leaf

we can also write Euler equations as

$$f y'' = f'(1+y'^2) \quad ; \quad f' = \frac{\partial f}{\partial y}$$

$$\Rightarrow \quad \frac{y' y''}{1+y'^2} = \frac{f'}{f} y' = \frac{d}{dx} \ln f$$

$$\Rightarrow \quad \frac{1}{2} \frac{\partial}{\partial y'} \ln(1+y'^2) * \frac{dy'}{dx} = \frac{1}{2} \frac{d}{dx} \ln(1+y'^2)$$

Thus $$\frac{d}{dx} \ln(1+y'^2)^{1/2} = \frac{d}{dx} \ln f$$

$$\Rightarrow \quad \ln \frac{(1+y'^2)^{1/2}}{f(y)} = c$$

$$\Rightarrow \quad (1+y'^2)^{1/2} = a f(y)$$

$$\Rightarrow \quad \boxed{1+y'^2 = A f^2(y)}$$

b/ For $f(y) = 1$ we get

$$J = \int ds = \int_{x_i}^{x_f} dx \sqrt{1+y'^2(x)}$$

J represent the length of the path connecting (x_i, y_i) and (x_f, y_f)
So we are looking for the path that minimize the distance between two points. In case of constant speed of light in a given medium

$$I = \int \frac{ds}{c} = \frac{1}{c}\int ds = \frac{J}{c}$$

Thus this integral also minimizes the time taken by light in propagation from $(x_i; y_0)$ to (x_f, y_f).

Euler Lagrange eq. gives

$$x = \pm \int \frac{dy}{\sqrt{A-1}} = cy + b$$

or in the usual form $\boxed{y = Ax + B}$ $\begin{cases} A = \frac{(y_f - y_0)}{(x_f - x_0)} \\ B = \frac{x_f y_0 - x_0 y_f}{(x_f - x_i)} \end{cases}$

The equation of a straight line

c/ $f(y) = \sqrt{y}$ then

$$x = \pm \int \frac{dy}{\sqrt{Ay-1}} = \pm A^{-1}\sqrt{Ay-1}$$

$$\Rightarrow \quad a x^2 = Ay - 1 \quad \Rightarrow \boxed{y = \alpha x^2 + \beta}$$

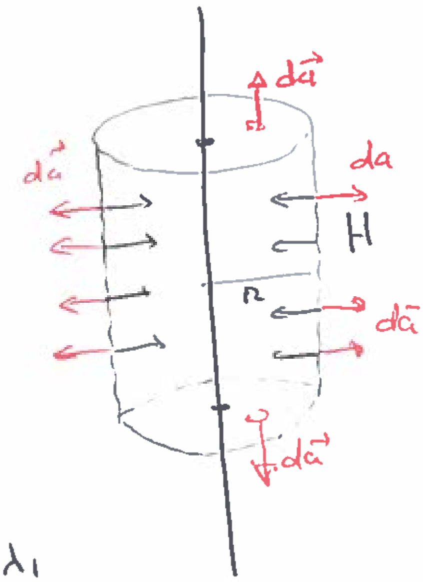
which is a parabolic path connecting (x_i, y_0) and (x_f, y_f),
one can then express α and β in terms of y_0, x_i and y_f, x_f through

$$\begin{cases} y_0 = \alpha x_0^2 + \beta \\ y_f = \alpha x_f^2 + \beta \end{cases}$$

$$\Rightarrow \quad \alpha = \left(\frac{y_f - y_i}{x_f^2 - x_i^2}\right) \quad ; \quad \beta = \left(\frac{y_f x_0 - y_i x_f}{x_i - x_f}\right)$$

Q2.

a/ By cylindrical symmetry $\vec{g}(r) = g(r)\,\hat{r}$



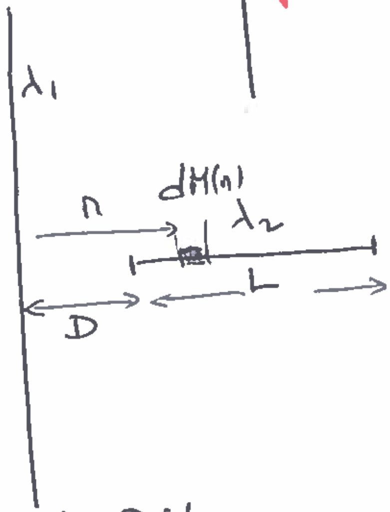
$$\oint \vec{g}\cdot d\vec{a} = -\int_S g(r)\,da = -g(r)(2\pi r H)$$

$$-g(r)(2\pi r \cancel{H}) = -4\pi G \lambda_1 \cancel{H}$$

$$\Rightarrow \quad g(r) = +\frac{2\lambda_1 G}{r}$$

$$\vec{g}(r) = -\frac{2\lambda_1 G}{r}\,\hat{r}$$

b/



$$d\vec{F} = +\,dM\,\vec{g}(r) = dM(r)\,\vec{g}(r)$$

$$dM(r) = \lambda_2\,dr \quad ; \quad \vec{g}(r) = -\frac{2\lambda_1 G}{r}\,\hat{r}$$

$$d\vec{F} = -\frac{2\lambda_1 G\,dM}{r}\,\hat{r}$$

$$d\vec{F} = -\frac{2\lambda_1 G \lambda_2\,dr}{r}\,\hat{r}$$

$$\vec{F} = -\hat{r}\int_D^{D+L} \frac{2\lambda_1\lambda_2 G\,dr}{r} = -\hat{r}\,(2\lambda_1\lambda_2 G)\ln\frac{D+L}{D}$$

$$\vec{F} = -\hat{r}\,(2\lambda_1\lambda_2 G)\ln\left(1+\frac{L}{D}\right)$$

c/ If $x = \frac{L}{D} \ll 1$ then $\ln(1+x) \simeq x$

$$\vec{F} \simeq -\hat{r}\,(2\lambda_1\lambda_2 G)\,x = -\hat{r}\left(2\lambda_1\lambda_2 G\,\frac{L}{D}\right) = -\hat{r}\left(2\lambda_1 G\,\frac{M}{D}\right)$$

$$\vec{F} \simeq -\hat{r}\,(g(D)\,M) = M\,\vec{g}(D)$$

Since $L \ll D$ then the second wire behave like a point mass M located a distance D from the infinite wire.

d/ $$\oint \vec{g}(r)\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{g}(r)\,dV = -4\pi G\int_V \rho\,dV \quad ; \quad \vec{g}(r) = -g(r)\,\hat{r}$$

thus $$\vec{\nabla}\cdot\vec{g}(r) = -4\pi G\rho \Rightarrow \rho(r) = -\frac{1}{4\pi G}\,\vec{\nabla}\cdot\vec{g}(r)$$

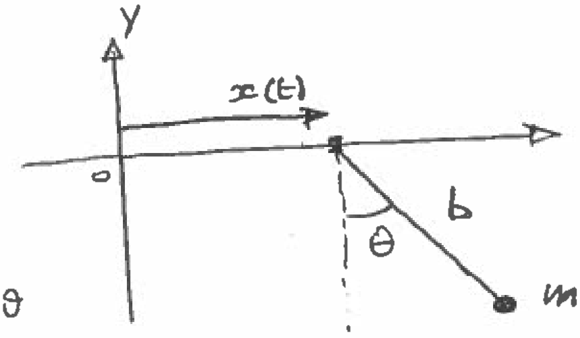
using the fact that in spherical coordinates

$$\vec{\nabla}\cdot\vec{g}(r) = -\frac{1}{r^2}\frac{d}{dr}\left(r^2 g(r)\right) \Rightarrow \rho(r) = +\frac{1}{4\pi r^2 G}\frac{d}{dr}\left(r^2 g(r)\right)$$

If $g(r) = \frac{a}{b^2+r^2} \Rightarrow \rho(r) = \frac{1}{4\pi G}\,\frac{2ab^2}{r\,(b^2+r^2)^2}$

Q3.

a/



$$L_2 = \frac{1}{2} m_2 (\dot{X}^2 + \dot{Y}^2) - m_2 g Y$$

$$X = x(t) + b\sin\theta \quad ; \quad Y = -b\cos\theta$$

$$\dot{X} = \dot{x}(t) + b\dot{\theta}\cos\theta \quad ; \quad \dot{Y} = b\dot{\theta}\sin\theta$$

$$L_2 = \frac{1}{2} m_2 [\dot{x}(t)^2 + (b\dot{\theta})^2 + 2\dot{x}\dot{\theta}\, b\cos\theta] + m_2 g b\cos\theta$$

$$L_1 = \frac{1}{2} m_1 \dot{x}(t)^2$$

$$L = L_1 + L_2 = \frac{1}{2}(m_1+m_2)\dot{x}(t)^2 + \frac{1}{2} m_2 b^2\dot{\theta}^2 + m_2 b\dot{x}\dot{\theta}\cos\theta + m_2 g b\cos\theta$$

Only two degrees of freedom because
(x_1, y_1) has a constant $y_1 = 0$, slides horizontally.
(X, Y) has a constraint $(X-x)^2 + Y^2 = b^2$, the length of the pendulum b is fixed.

Thus $2\times 2 - 2 = 4 - 2 = 2$ degrees of freedom.

b/

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad ; \quad \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

x is cyclic : $\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x} + m_2 b\dot{\theta}\cos\theta = \text{Cstat}$

$$(m_1+m_2)\dot{x} + m_2 b\dot{\theta}\cos\theta = P_x$$

expresses the conservation of the linear momentum along the x-direction

since $F_x = 0$ $\quad F_x = \frac{dP_x}{dt} = 0 \Rightarrow P_x = \text{Cstat.}$

as $U = U(y)$ $\Rightarrow P_x = \text{Cstat.}$

$$\frac{\partial L}{\partial \theta} = -m_2 b\dot{x}\dot{\theta}\sin\theta - m_2 g b\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}[m_2 b^2\dot{\theta} + m_2 b\dot{x}\cos\theta] = m_2 b^2\ddot{\theta} + m_2 b\ddot{x}\cos\theta - m_2 b\dot{\theta}\dot{x}\sin\theta$$

$$m_2 b^2\ddot{\theta} + m_2 b\ddot{x}\cos\theta - m_2 b\dot{\theta}\dot{x}\sin\theta + m_2 b\dot{\theta}\dot{x}\sin\theta + m_2 g b\sin\theta = 0$$

$$\ddot{\theta} + \frac{1}{b}\ddot{x}\cos\theta - \cancel{\frac{1}{b}\dot{\theta}\dot{x}\sin\theta} + \cancel{\frac{1}{b}\dot{\theta}\dot{x}\sin\theta} + \frac{g}{b}\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \omega_0^2\sin\theta = -\frac{\ddot{x}}{b}\cos\theta} \quad ; \quad \omega_0^2 = \frac{g}{b}$$

c/ $x(t) = A\cos\omega t$ then $\ddot{x}(t) = -\omega^2 A\cos\omega t$

Then our previous equation of motion for θ becomes

$$\ddot{\theta} + \omega_0^2 \sin\theta = \frac{\omega^2 A}{b}\cos\omega t\,\cos\theta$$

For small angle oscillations

$$\sin\theta \simeq \theta \quad ; \quad \cos\theta \simeq 1$$

$$\Rightarrow \quad \ddot{\theta} + \omega_0^2\,\theta = \frac{\omega^2 A}{b}\cos\omega t = \tilde{A}\cos\omega t \,; \quad \tilde{A} = \frac{\omega^2 A}{b}$$

Comparing with (3.53) and (3.60) gives

$$x_p(t) = \frac{\tilde{A}}{|\omega_0^2 - \omega^2|}\cos\omega t \quad ; \quad x_h(t) = B\cos(\omega_0 t + \beta)$$

Since $\delta = \tan^{-1}(0) = 0$, $\beta = 0$ in (3.53). The most general solution is

$$x(t) = x_h(t) + x_p(t) = B\cos(\omega_0 t + \beta) + \frac{\omega^2 A/b}{|\omega_0^2 - \omega^2|}\cos\omega t$$

This solution diverges as $\omega \to \omega_0$, a resonance phenomenon occurs.