Q1.

a/ A is a proper rotation if $\det A = |A| = 1$

$$|A| = \begin{vmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{vmatrix} = \begin{vmatrix} 1 & -1/3 & 2/3 \\ 1 & 2/3 & -1/3 \\ 1 & 2/3 & 2/3 \end{vmatrix} = \begin{vmatrix} 1 & -1/3 & 2/3 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$ thus A is a proper rotation.

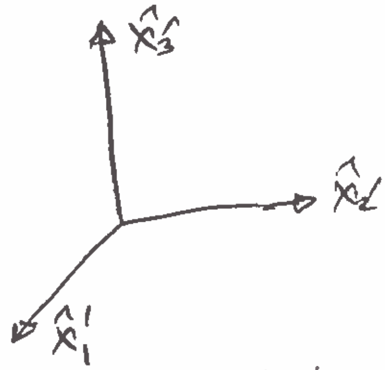
b/ Show that $AV = V$; $V = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$AV = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = V$$

thus V is invariant under rotation A $\Rightarrow$ V is the axis of rotation.

c/ Assume that $\hat{v} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \hat{x}_3'$ while $\hat{x}_1', \hat{x}_2'$ are $\perp \hat{x}_3'$ basis $(\hat{x}_1', \hat{x}_2', \hat{x}_3')$ is the rotation matrix in this

$$A' = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



c/ If we transform $P_1 = (0, 4, 0)$ by A then

$$P_1' = A P_1 = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 8/3 \\ 8/3 \end{pmatrix} = \frac{4}{3}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$P_2' = A P_2 = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$$

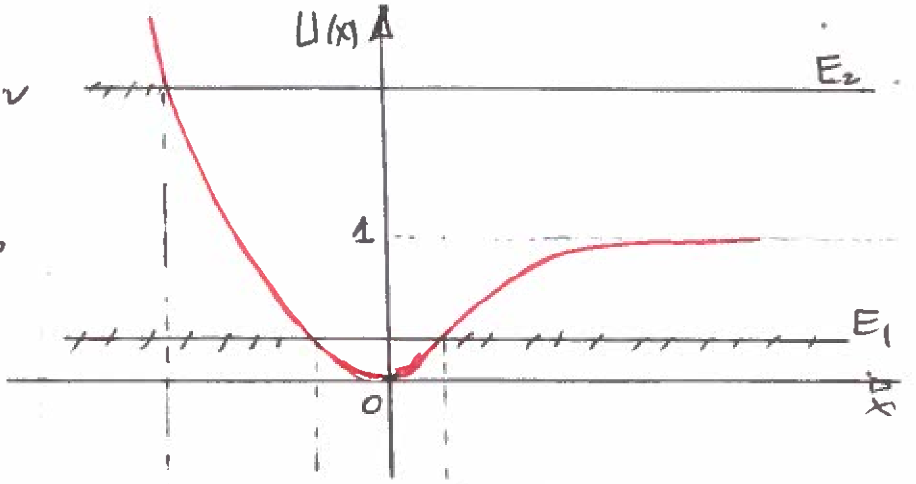
d/

$$P_2'P_1' = |P_2' - P_1'| = \left|\begin{pmatrix} 10/3 \\ -2/3 \\ -11/3 \end{pmatrix}\right| = \frac{1}{3}\sqrt{10^2 + 2^2 + 11^2} = 5 = P_2 P_1$$

distances between points are not affected by rotations.

Q2.

a) Sketch $U(x) = (e^{-\alpha x} - 1)^2$



$$U(x) \simeq \begin{cases} 1 & \text{as } x \to +\infty \\ e^{-2\alpha x} & \text{as } x \to -\infty \end{cases}$$

$U(0) = 0$

$$\frac{dU}{dx} = 2(-\alpha e^{-\alpha x})(e^{-\alpha x} - 1)$$

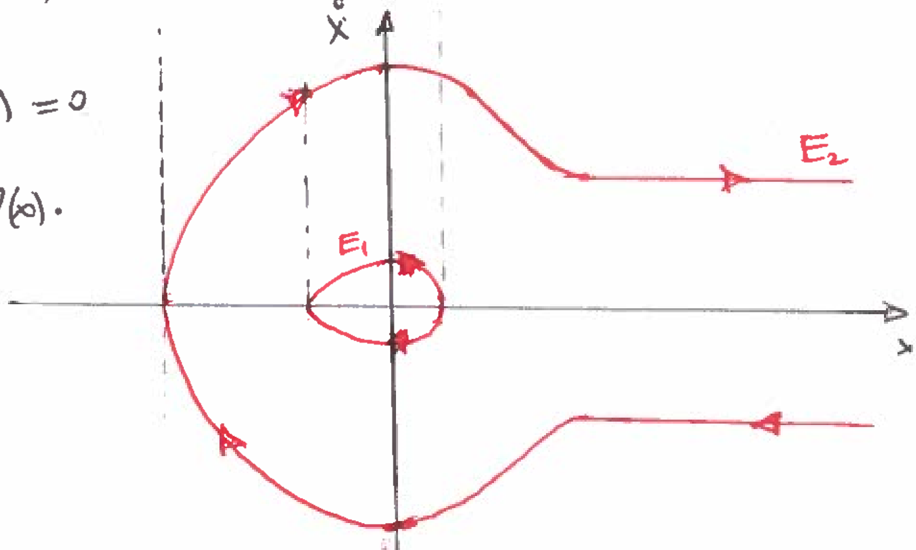
$$\frac{dU}{dx} = 0 \Rightarrow e^{-\alpha x} = 1 \Rightarrow x = 0$$

$$\frac{d^2U}{dx^2} = e^{-\alpha x}\left[2\alpha^2(e^{-\alpha x} - 1) + 2\alpha^2 e^{-\alpha x}\right] = 2\alpha^2 e^{-\alpha x}(2e^{-\alpha x} - 1)$$

$$\left.\frac{d^2U}{dx^2}\right|_{x=0} = 2\alpha^2 > 0 \ ; \quad U(0) = 0$$

Thus $x=0$ is a minimum for $U(x)$.

b)



c) For $E_1 = \varepsilon \ll 1$ the particle is very close to $x=0$ so we can expand

$$U(x) \simeq \cancel{U(0)} + \cancel{x \left.\frac{dU}{dx}\right|_0} + \frac{1}{2}x^2 \left.\frac{d^2U}{dx^2}\right|_0 = \frac{1}{2}(2\alpha^2)\, x^2 = \frac{1}{2}kx^2$$

$$\text{thus} \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{2\alpha^2}} = \frac{\pi}{\alpha}\sqrt{2m}$$

Q3.

a) $$m\frac{dV}{dt} = -\alpha e^{\beta V} \Rightarrow \int_{V_0}^{V} e^{-\beta V} dV = -\int_0^t \frac{\alpha}{m} dt$$

$$\Rightarrow -\frac{\alpha}{m}t = \frac{1}{\beta}\left(e^{-\beta V_0} - e^{-\beta V}\right)$$

$$\Rightarrow e^{-\beta V} = e^{-\beta V_0} + \frac{\alpha\beta}{m}t$$

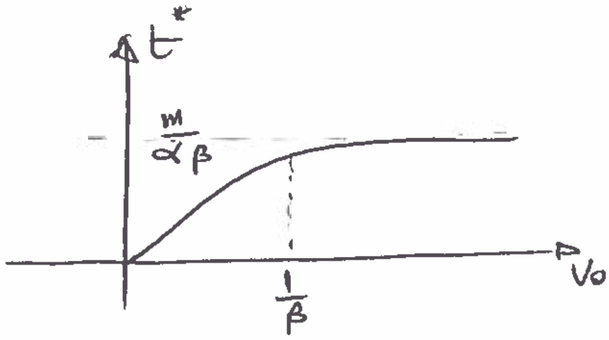
$$\Rightarrow V(t) = -\frac{1}{\beta}\ln\left[\frac{\alpha\beta}{m}t + e^{-\beta V_0}\right]$$

b) To stop we need $V = 0$

$$\Rightarrow \ln\left[\frac{\alpha\beta}{m} t^* + e^{-\beta V_0}\right] = 0$$

$$\Rightarrow \frac{\alpha\beta}{m} t^* + e^{-\beta V_0} = 1$$

$$\Rightarrow t^* = \frac{m}{\alpha\beta}\left(1 - e^{-\beta V_0}\right)$$



c) $K = \frac{1}{2} m V^2$

$$\frac{dK}{dt} = m V \frac{dV}{dt} = V\left(-\alpha e^{\beta V}\right) < 0 \quad \text{since } \alpha, V, \beta > 0$$

Q4.

a) $\ddot{x} + 8\dot{x} + 41 x = 0$ let $x = e^{rt}$

$$\rightarrow (r^2 + 8r + 41) e^{rt} = 0$$

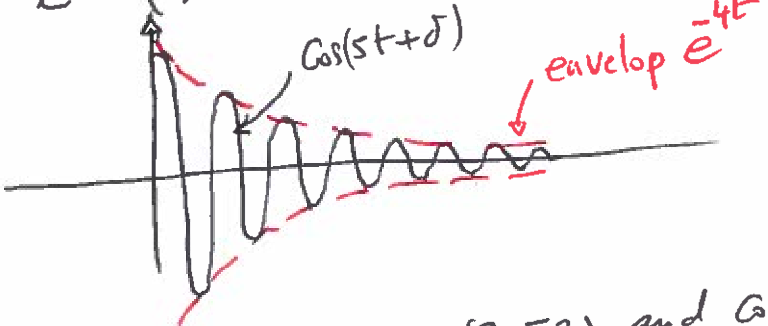
$$\Rightarrow r = -4 \pm \sqrt{4^2 - 41} = -4 \pm i5$$

$$x = Re\{A e^{rt}\} = Re\{A e^{(-4t \pm i5t)}\} = Re\{A e^{-4t} e^{\pm i5t}\}$$

$$x = e^{-4t} |A| \cos(5t + \delta)$$

or $x = \frac{1}{2} e^{-4t}\left(A e^{i5t} + A^* e^{-i5t}\right) = |A| e^{-4t} \cos(5t+\delta)$

b)



Since it is open book then let us use (3.53) and compare it to our

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A \cos\omega t$$

then $2\beta = 8 \Rightarrow \beta = 4$; $\omega_0^2 = 41$; $A = 20$; $\omega = 6$

So that $$D = \frac{20}{\sqrt{(41-6^2)^2 + 4 \times 6^2 \times 4^2}} = \frac{20}{\sqrt{2329}} = 0.414 \approx 0.41$$

Since $\omega_0^2 = 41 > \beta^2 = 4^2 = 16 \Rightarrow$ {We will be in underdamped regime,

$$\delta = \tan^{-1}\left(\frac{2(6)4}{41-36}\right) = 84^\circ = \frac{\pi \times 84}{180} = 1.465 \simeq 1.5$$

Thus $X_p(t) = 0.41 \cos(6t - 1.5)$ (3.55)

c/ The most general solution is

$$X(t) = X_h(t) + X_p(t)$$
$$= 0.41 \cos(6t - 1.5) + A\, e^{-4t} \cos(5t + \delta)$$

A, δ will be fixed by the initial condition

$$X_0 = X(0) = 0.41 \cos(-1.5) + A \cos\delta$$
$$V_0 = \dot{X}(0) = 0.41 * 6 \sin(+1.5) - 5A \sin\delta$$

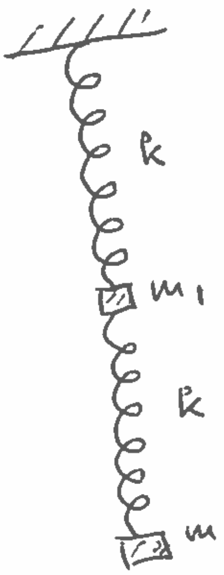
$$\Rightarrow \quad A\cos\delta = X_0 - 0.043$$
$$A \sin\delta = (2.45 - V_0)/5 = 0.49 - 0.2V_0$$

$$A = \sqrt{(X_0 - 0.043)^2 + (0.49 - 0.2V_0)^2}$$

$$\tan\delta = \left(\frac{0.49 - 0.2V_0}{X_0 - 0.043}\right)$$

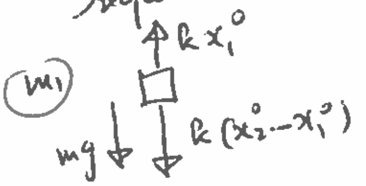
d) X_h represents the transient solution and decay as e^{-4t} that in a time scale of $\frac{1}{4} = 0.25$ s.

X_p is the steady state oscillatory solution

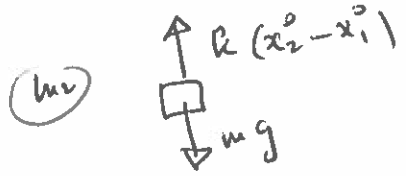


Q5.

a/ Let us call x_1^0 and x_2^0 the streching of each spring, then static equilibrium requires



$$mg + k(x_2^0 - x_1^0) = k x_1^0 \quad (1)$$
$$k(x_2^0 - x_1^0) = mg \quad (2)$$



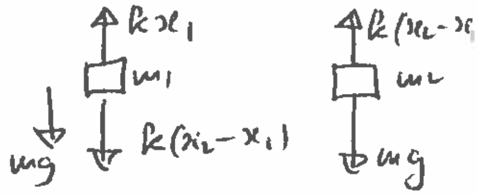
(1) $\Rightarrow$ $mg + mg = kx_1^0 \Rightarrow x_1^0 = \frac{2mg}{k}$ (3)

(2) $\Rightarrow$ $x_2^0 = x_1^0 + \frac{mg}{k} = \frac{3mg}{k}$ (4)

b) Now let x_1, x_2 be the displacement of m_1, m_2 from their unstretched positions, then the equations of motion are (5)

$$(m_1) \quad m\ddot{x}_1 = mg + k(x_2 - x_1) - kx_1 \quad (5)$$

$$(m_2) \quad m\ddot{x}_2 = mg - k(x_2 - x_1) \quad (6)$$



c) Let $X_1 = x_1 - x_1^0$; $X_2 = x_2 - x_2^0$

$\Rightarrow \dot{X}_1 = \dot{x}_1$, $\ddot{X}_1 = \ddot{x}_1$, $\dot{X}_2 = \dot{x}_2$, $\ddot{X}_2 = \ddot{x}_2$

$$(5) \Rightarrow m\ddot{X}_1 = mg + k(x_1^0 - x_2^0 + X_2 - X_1) - k(X_1 + x_1^0)$$

$$= mg + k(x_1^0 - x_2^0) - kx_1^0 + k(X_2 - X_1) - kX_1$$

$$m\ddot{X}_1 = k(X_2 - X_1) - kX_1 \quad \text{using (1)}$$

$$(6) \Rightarrow m\ddot{X}_2 = mg - k(x_1^0 - x_2^0) - k(X_2 - X_1)$$

$$m\ddot{X}_2 = -k(X_2 - X_1) \quad \text{using (2)}$$

Thus

$$\begin{cases} \ddot{X}_1 = \omega_0^2 [X_2 - X_1 - X_1] = \omega_0^2 [X_2 - 2X_1] & (7) \\ \ddot{X}_2 = -\omega_0^2 (X_2 - X_1) \quad ; \quad \omega_0^2 = \frac{k}{m} & (8) \end{cases}$$

d)

X_1, X_2 represent the displacements of m_1 and m_2 from their equilibrium positions defined in (a). The equations can be written in matrix form

$$\frac{d^2}{dt^2}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \omega_0^2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad (9)$$

We look for eigenmodes $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i\omega t}$ (10)

$$(9) \Rightarrow \begin{pmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega^2 - \omega_0^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (11)$$

$$\Rightarrow \begin{vmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega^2 - \omega_0^2 \end{vmatrix} = 0 \Rightarrow (\omega^2 - 2\omega_0^2)(\omega^2 - \omega_0^2) - \omega_0^4 = 0$$

$$\omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0$$

$$\Rightarrow \quad \omega^2 = \frac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2} = \omega_0^2\left(\frac{3\pm\sqrt{5}}{2}\right)$$

Thus two eigenmodes:

$$\omega_1^2 = \omega_0^2\left(\frac{3+\sqrt{5}}{2}\right) \quad ; \quad \omega_2^2 = \omega_0^2\left(\frac{3-\sqrt{5}}{2}\right)$$

a $\quad \omega_1 = \omega_0\sqrt{\frac{3+\sqrt{5}}{2}} \quad ; \quad \omega_2 = \omega_0\sqrt{\frac{3+\sqrt{5}}{2}}$

Eigenvectors:

For $\omega = \omega_1$ the first equation in (1)

$$(\omega^2 - 2\omega_0^2)A_1 = -\omega_0^2 A_2$$

gives

$$\cancel{\omega_0^2}\left(\frac{3-\sqrt{5}}{2} - 2\right)A_1 = -\cancel{\omega_0^2}A_2$$

$$\left(\frac{1+\sqrt{5}}{2}\right)A_1 = A_2$$

$$\Rightarrow \quad V_1 = A_1\begin{pmatrix}1\\ \frac{\sqrt{5}+1}{2}\end{pmatrix}e^{i\omega_1 t}$$

For $\omega = \omega_2$ the above equation gives

$$\cancel{\omega_0^2}\left(\frac{3+\sqrt{5}}{2} - 2\right)A_1 = -\cancel{\omega_0^2}A_2$$

$$\left(\frac{1-\sqrt{5}}{2}\right)A_1 = A_2$$

$$\Rightarrow \quad A_2 = \left(\frac{1-\sqrt{5}}{2}\right)A_1$$

thus $\quad V_2 = A_1\begin{pmatrix}1\\ \frac{1-\sqrt{5}}{2}\end{pmatrix}e^{i\omega_2 t}$

In the first mode the two mass move in the same direction with different amplitude while in the second mode they move in opposite directions.

e) The most general solution is a linear combination of the above two modes

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{\sqrt{5}+1}{2} \end{pmatrix} e^{i\omega_1 t} + c_2 \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} e^{i\omega_2 t}$$

Where in general $c_1 = |c_1| e^{i\alpha}$ and $c_2 = |c_2| e^{i\beta}$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = |c_1| \begin{pmatrix} 1 \\ \frac{\sqrt{5}+1}{2} \end{pmatrix} e^{i(\omega_1 t+\alpha)} + |c_2| \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix} e^{i(\omega_2 t+\beta)}$$

Taking the real parts:

$$\begin{cases} x_1(t) = |c_1| \cos(\omega_1 t+\alpha) + |c_2| \cos(\omega_2 t+\beta) \\ x_2(t) = |c_1| \left(\frac{\sqrt{5}+1}{2}\right) \cos(\omega_1 t+\alpha) + |c_2| \left(\frac{1-\sqrt{5}}{2}\right) \cos(\omega_2 t+\beta) \end{cases}$$

Answers to Missing Questions.

Q1.a/

$$AA^t = \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{pmatrix}$$

$$AA^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{1} \Rightarrow A^{-1} = A^t = \begin{pmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \end{pmatrix}$$

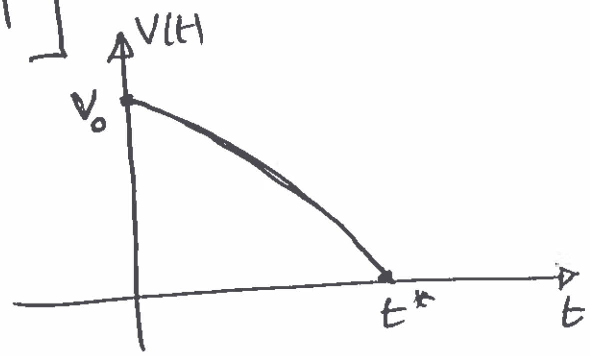
Q3.c/ Write $V(t)$ in terms of t and t^*

$$V(t) = -\frac{1}{\beta} \ln\left[\frac{\alpha\beta}{m} t + e^{-\beta V_0}\right]$$

from the equation defining t^* we have

$$t^* = \frac{m}{\alpha\beta}\left(1 - e^{-\beta V_0}\right) \Rightarrow e^{-\beta V_0} = 1 - \frac{\alpha\beta}{m} t^*$$

$$\Rightarrow \quad V(t) = -\frac{1}{\beta} \ln\left[\frac{\alpha\beta}{m}(t - t^*) + 1\right]$$



one can show that

$$\left.\frac{dV}{dt}\right|_0 = -\frac{\alpha}{m} e^{\beta V_0} < 0$$

$$\left.\frac{dV}{dt}\right|_{t^*} = -\frac{\alpha}{m} < 0$$

and extremum exists for $V(t)$ in t $[0, t^*]$ interval.