KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

DEPARTMENT OF PHYSICS

PHYS.300- Classical Mechanics I (TERM 211)

Quiz #4

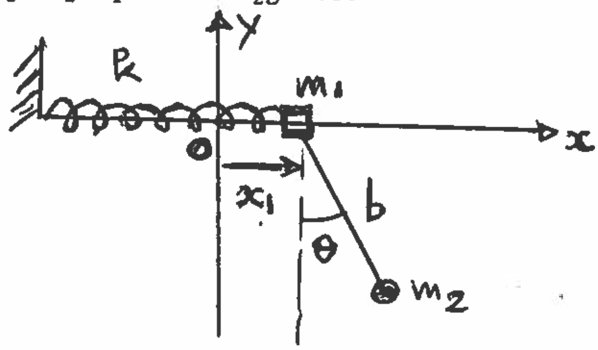
Name:

ID #

Q. Consider a point masse $\mathbf{m_1}$ attached to the end of a spring having spring constant **k** and fixed to the wall at one end. The mass $\mathbf{m_1}$ moves on a frictionless horizontal table, x_1 is its displacement from equilibrium point **O** (un-stretched spring). A simple pendulum of length **b** is attached to $\mathbf{m_1}$, has a point mass $\mathbf{m_2}$ at its end and is free to oscillate in a **vertical plane**.

a) Show that the **Lagrangian** of the system can be written in the following form:

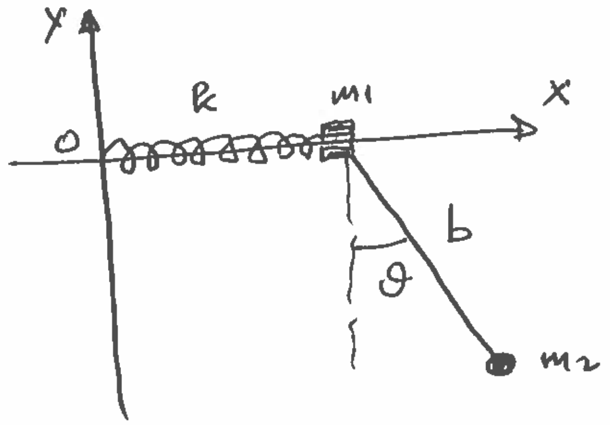
$L(x_1, \theta; t) = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2}m_2 b^2\dot{\theta}^2 - \frac{1}{2}kx_1^2 - m_2 b\ddot{x}_1 \sin\theta + m_2 gb\cos\theta$



b) Write down the **Lagrange equations** of motion and **solve them** explicitly.

Quiz # 4

Chap. 7 Solution



a) Lagrangian

The system has two degrees of freedom x_1 the displacement of m_1 from the unstretched position of the spring and θ for the simple pendulum

$$L = T - U$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \quad ; \quad U = m_2 g y_2$$

(x_1) is the coordinate for m_1, (x_2, y_2) are the coordinates for m_2.

$$\begin{cases} x_2 = x_1 + b \sin\theta \\ y_2 = -b\cos\theta \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = \dot{x}_1 + b\dot{\theta}\cos\theta \\ \dot{y}_2 = b\dot{\theta}\sin\theta \end{cases}$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 [\dot{x}_1^2 + (b\dot{\theta})^2 + 2b\dot{\theta}\dot{x}_1\cos\theta]$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2} m_2 b^2 \dot{\theta}^2 + m_2 b \dot{\theta} \dot{x}_1 \cos\theta$$

$$U_g = m_2 g y_2 = -m_2 g b \cos\theta \quad ; \quad U_s = \frac{1}{2} k x_1^2$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2} m_2 b^2 \dot{\theta}^2 + \frac{d}{dt}(m_2 b \dot{x}_1 \sin\theta) - m_2 b \ddot{x}_1 \sin\theta + m_2 g b \cos\theta - \frac{1}{2} k x_1^2$$

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}_1^2 + \frac{1}{2} m_2 b^2 \dot{\theta}^2 - m_2 b \ddot{x}_1 \sin\theta + m_2 g b \cos\theta - \frac{1}{2} k x_1^2$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad ; \quad \frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = 0$$

$$\begin{cases} -k x_1 - (m_1 + m_2)\ddot{x}_1 = 0 \\ -m_2 b \ddot{x}_1 \cos\theta - m_2 g b \sin\theta - \frac{d}{dt}[m_2 b^2 \dot{\theta}] = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{x}_1 + \omega_0^2 x_1 = 0 \quad ; \quad \omega_0^2 = \frac{k}{m_1 + m_2} \\ \ddot{\theta} + \Omega^2 \sin\theta + \frac{1}{b}\ddot{x}_1 \cos\theta = 0 \quad ; \quad \Omega^2 = \frac{g}{b} \end{cases}$$

For small oscillation x_1 and θ are very small so that

$$\sin\theta \simeq \theta \quad ; \quad \cos\theta \simeq 1$$

$$\ddot{x}_1 + \omega_0^2 x_1 = 0$$

$$\Rightarrow x_1 = A \cos(\omega_0 t + \delta)$$

Then $\ddot{\theta} + \Omega^2 \theta \simeq -\frac{\ddot{x}_1}{b} \simeq \frac{A}{b} \omega_0^2 \cos(\omega_0 t + \delta_1)$

use the solution of (3.53) with $\beta = 0$; $\Omega = \omega_0$ and $\omega = \omega_0$

the solution is given by (3.60) and (3.61) in our case

$$\theta = B \cos(\Omega t + \alpha) + \frac{(A\omega_0^2/b)}{|\omega^2 - \Omega^2|} \cos(\omega_0 t + \delta_1)$$

$$\delta_1 = \tan^{-1}(0) = 0$$

using the initial conditions:

$$\begin{cases} x(0) = x_0 & ; \quad \dot{x}_1(0) = 0 \\ \theta(0) = \theta_0 & ; \quad \dot{\theta}(0) = 0 \end{cases}$$

gives
$$x_1(0) = A \cos\delta = x_0 = A \Rightarrow A = x_0$$
$$\dot{x}_1(0) = -\omega_0 A \sin\delta = 0 \Rightarrow \delta = 0$$

thus $x_1(t) = x_0 \cos(\omega_0 t)$

$$\theta(t) = B \cos(\Omega t + \alpha) + \frac{x_0 \omega_0^2/b}{|\omega^2 - \Omega^2|} \cos(\omega_0 t)$$

$$\theta(0) = B \cos\alpha + \frac{x_0 \omega_0^2 b}{|\omega^2 - \Omega^2|} = \theta_0$$

$$\dot{\theta}(t) = -\Omega A \sin(\Omega t + \alpha) - \frac{x_0 \omega_0^3/b}{|\omega^2 - \Omega^2|} \sin(\omega_0 t)$$

$$\dot{\theta}(0) = -\Omega A \sin\alpha - \cancel{\frac{x_0 \omega_0^3/b}{|\omega^2 - \Omega^2|}} \times 0 = 0$$

$$\Rightarrow \sin\alpha = 0 \Rightarrow \boxed{\alpha = 0}$$

and $\theta_0 = B + \frac{x_0 \omega_0^2 b}{|\omega^2 - \Omega^2|} = \theta_0 \Rightarrow \boxed{B = \theta_0 - \frac{\omega_0^2 x_0/b}{|\omega^2 - \Omega^2|}}$

$$\theta(t) = \left(\theta_0 - \frac{\omega_0^2 x_0/b}{|\omega^2 - \Omega^2|}\right) \cos\Omega t - \frac{x_0 \omega_0^2/b}{|\omega^2 - \Omega^2|} \cos\omega_0 t$$

$$x_1(t) = x_0 \cos(\omega_0 t)$$