King Fahd University of Petroleum and Minerals – Physics Department
PHYS305 – Electricity and Magnetism I – Term 211 – Fall 2021
Second Major exam 05 December, 2021 (Time 2:00 hours)

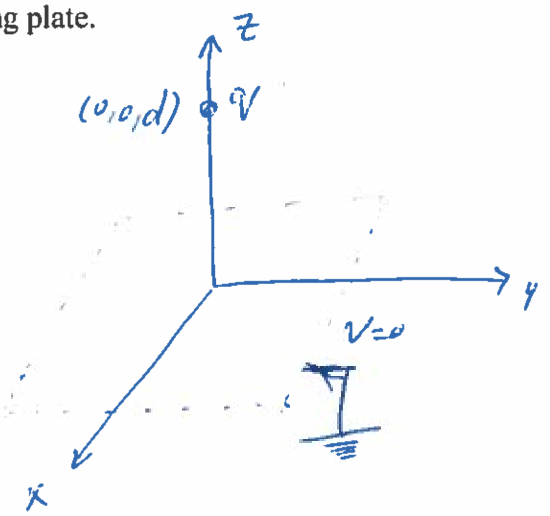
Q#1: (5+5+5+5)

Consider a charge q is placed on z-axis at a distance d above an infinite grounded conducting plate centered at the origin in the *xy-plane*.

a) Find the potential due to this charge configuration at point P above the *xy-plane*.
b) Find the electric field at point P.
c) Find the surface charge distribution on the conducting plate.
d) Find the force between the charge and the conducting plate.

a)

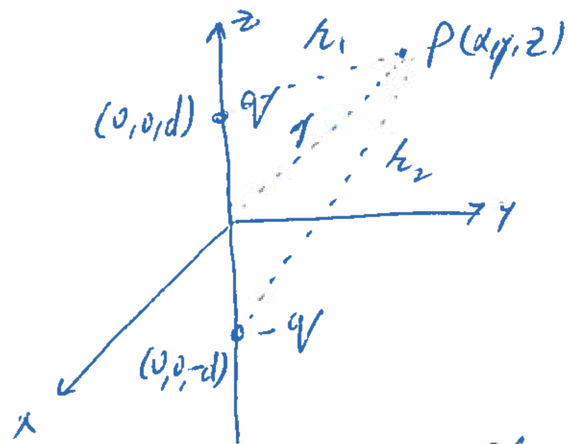
The boundary conditions for this problem are:



$$V(x,y,0) = 0$$

$$V(x,y,z) = 0 \quad \text{when } x \to \infty,\ y \to \infty,\ z \to \infty$$

An image charge $-q$ at distance $-d$ along the z-axis will satisfy the same boundary conditions.



So potential in the space $z>0$ is

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1} + \frac{-q}{r_2}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right)$$

b)

$$E(x,y,z) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1^2} + \frac{-q}{r_2^2}\right) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{[x^2+y^2+(z-d)^2]} - \frac{q}{(x^2+y^2+(z+d)^2)}\right]$$

(c)

$$\bar{E}_{outside} = \frac{\sigma}{\epsilon_0}\hat{n} \Rightarrow \sigma = \epsilon_0 E_z\Big|_{z=0} = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$$

Q# 1 Continued

(c)

$$\sigma = -\epsilon_0 \left[\frac{q}{4\pi\epsilon_0} \left\{ \frac{-(z-d)}{\left(x^2+y^2+(z-d)^2\right)^{3/2}} + \frac{(z+d)}{\left(x^2+y^2+(z+d)^2\right)^{3/2}} \right\} \right]_{z=0}$$

$$\sigma = \frac{-q}{4\pi} \left[\frac{-z+d+z+d}{\left(x^2+y^2+d^2\right)^{3/2}} \right] = \frac{-q}{2\pi} \frac{d}{\left(x^2+y^2+d^2\right)^{3/2}}$$

$$\boxed{\sigma = -\frac{q}{2\pi} \frac{d}{\left(x^2+y^2+d^2\right)^{3/2}}}$$

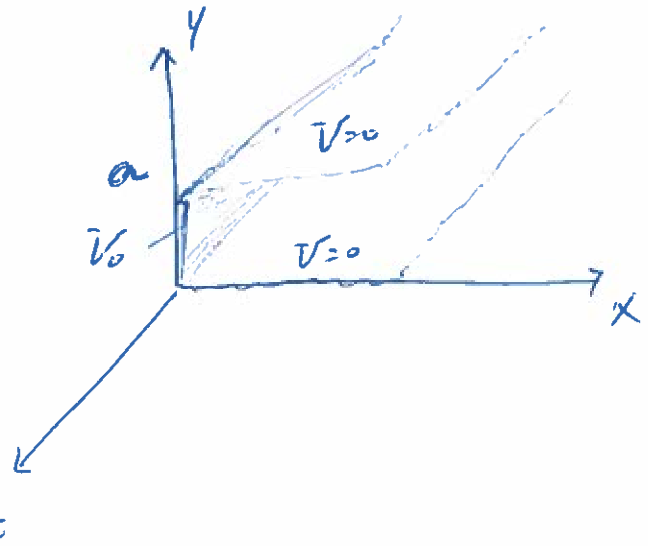
(d) The force between q and grounded Conducting Plate is the Same as the force between the charge q and the image charge $-q$. The force on charge q is:

$$\bar{F} = \frac{1}{4\pi\epsilon_0} \frac{q\times(-q)}{(2d)^2} \hat{h} = \boxed{-\frac{1}{16\pi\epsilon_0} \frac{q^2}{d^2}}$$

Q#2: (5+5)

Two infinite grounded metal plates lie parallel to the *xz-plane* one at $y=0$ and the other at $y=a$. The left end at $x=0$ is closed off with an infinite metal strip insulated from the two plates and maintained at a specific potential V_o.

(a) Find the potential inside the slot created by the metal plates and the strip.
(b) Determine the charge density on the metal strip.



Since there is no charge in the slot, so Laplacian equation in 2D can be written as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

and the boundary conditions are:

1- $V(x, y=0) = 0$

2- $V(x, y=a) = 0$

3- $V(x=0, y) = V_o$

4- $V \to 0$ when $x \to \infty$

using separation of variable method to solve Laplace's eq.

$$V(x,y) = X(x)\, Y(y)$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1 \quad \text{and} \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -C_1$$

Q# 2(a) Continued

let $C_1 = k^2 > 0$

$$\frac{d^2X}{dx^2} = k^2 X(x) \Rightarrow X(x) = Ae^{kx} + Be^{-kx}$$

B.C. 4 implies that $X(x) \to 0$ when $x \to \infty$ which means $A = 0$

so $\boxed{X(x) = Be^{-kx}}$

and $Y(y) = C \sin(ky) + D \cos(ky)$

hence $V(x,y) = X(x)Y(y) = e^{-kx}\left(C \sin(ky) + D \cos(ky)\right)$

B.C. 1
at $y = 0$ $V(x,y) = 0$

$$0 = e^{-kx}(0 + D) \Rightarrow D = 0$$

$$V(x,y) = Ce^{-kx} \sin(ky)$$

B.C. 2 $V(x,a) = 0 \Rightarrow Ce^{-kx} \sin(ka) = 0 \Rightarrow ka = n\pi$

$\boxed{k = \frac{n\pi}{a}}$

$n = 1, 2, 3, \ldots$

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a} y\right)$$

B.C. 3 $V(0,y) = V_0 \Rightarrow V_0 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} y\right)$

integrating both sides after multiplying by $\sin\left(\frac{n'\pi}{a} y\right)$

Q# 2(a) Continued

$$\int_0^a V_0 \sin\left(\frac{n'\pi y}{a}\right) dy = \sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n'\pi}{a} y\right) dy$$

$$\frac{-V_0 \cos\left(\frac{n\pi}{a} y\right)\Big|_0^a}{\left(\frac{n\pi}{a}\right)} = C_n \times \begin{cases} 0 & \text{for } n' \neq n \\ \frac{a}{2} & \text{for } n' = n \end{cases}$$

for $n' = n$

$$-\frac{aV_0}{n\pi}\Big(\cos(n\pi) - \cos(0)\Big) = C_n \times \frac{a}{2}$$

for $n = \text{odd}$

$$C_n = \frac{4V_0}{n\pi}$$

for $n = \text{even}$

$$C_n = 0$$

$$V(x,y) = \sum_{n=1,3,5,\cdots} C_n e^{-\frac{n\pi}{a} x} \sin\left(\frac{n\pi}{a} y\right)$$

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\cdots} \frac{1}{n} e^{-\frac{n\pi}{a} x} \sin\left(\frac{n\pi}{a} y\right)}$$

(b) at $x=0$ the boundary condition for the electric field is

$$E_{x=0_+} - E_{x=0_-} = E_{x=0_+} = \frac{\sigma}{\epsilon_0} \hat{n}$$

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial x}\Big|_{x=0} = -\epsilon_0 \frac{4V_0}{\pi} \sum_{n=1,3,5\cdots} \frac{1}{n}\left(-\frac{n\pi}{a}\right) e^{-\frac{n\pi}{a} x} \sin\left(\frac{n\pi}{a} y\right)$$

Q#2(b) Continued

$$\boxed{\sigma = -\epsilon_0 \frac{\partial V}{\partial x}\Big|_{x=0} = \frac{4 V_0 \epsilon_0}{a} \sum_{n=1,3,5\ldots} \sin\left(\frac{n\pi}{a} y\right)}$$

Q#3: (5+5)

A surface charge density $\sigma(\theta) = k\cos\theta$, where k is a constant is glued over a sphere of radius R. Find the potential inside and outside the sphere.

The general solution for Laplace's equation in Spherical Coordinates is:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

outside

at $r = \infty$ $V(r,\theta) \to 0$ this implies that $A_l = 0$

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \quad \text{— (1)}$$

Inside

inside the sphere at $r=0$ $V \to \infty$ if $B_l \neq 0$ so

$B_l = 0$

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \quad \text{— (2)}$$

at $r = R$ the two potentials should be equal, so

(1) = (2)

$$\sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$$

$$\Rightarrow B_l = A_l R^{2l+1}$$

Also the surface charge density $\sigma = -\epsilon_0\left(\frac{\partial V}{\partial r}\Big|_{r=R,\,out} - \frac{\partial V}{\partial r}\Big|_{r=R,\,in}\right)$

Q#3 Continued

$$\sigma = -\epsilon_0 \left(-\sum_{l=0}^{\infty} (l+1) \frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) \right)$$

Substituting $B_l = A_l R^{2l+1}$

$$-\frac{\sigma}{\epsilon_0} = +\left[\sum_{l=0}^{\infty} \left[\frac{(l+1)}{R^{l+2}} A_l R^{2l+1} + l A_l R^{l-1} \right] P_l(\cos\theta) \right.$$

$$\frac{\sigma(\theta)}{\epsilon_0} = \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta)$$

$$\frac{k}{\epsilon_0} \cos\theta = \sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta)$$

this implies that only A_1 is non-zero $A_2, A_3, A_4 \cdots A_\infty = 0$

$$\frac{k}{\epsilon_0} \cos\theta = 3 A_1 P_1(\cos\theta) = 3A_1 \cos\theta$$

$$\boxed{A_1 = \frac{k}{3\epsilon_0}}$$

outside

$$V(r,\theta) = \sum_{l=0}^{\infty} B_l \frac{1}{r^{l+1}} P_l(\cos\theta) = \sum_{l=0}^{\infty} A_l \frac{R^{2l+1}}{r^{l+1}} P_l(\cos\theta)$$

$$V_{out} = (r,\theta) = A_1 \frac{R^3}{r^2} \cos\theta = \boxed{\frac{k}{3\epsilon_0} \frac{R^3}{r^2} \cos\theta}$$

inside

$$V_{in} = A_1 r P_1(\cos\theta) = \frac{k}{3\epsilon_0} r \cos\theta$$

Q#4: (5+5)

A sphere of radius R carries a polarization $\vec{P} = k\vec{r}$, where k is a constant and $\vec{r}$ is the vector from the center.

(a) Calculate the bound surface and volume charge.
(b) Find the electric field inside and outside the sphere.

a) $$\sigma_b = \bar{P}\cdot\hat{n}\Big|_{r=R} = k\,\bar{r}\cdot\hat{r}\Big|_{r=R} = kR$$

$$\rho_b = -\bar{\nabla}\cdot\bar{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k r\right) = -3k$$

b) Electric field outside the Sphere due to Surface Charge

$$E_{out_{Surf}} = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma\, da}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{kR}{r^2}\times\int_{Surface} da = \frac{1}{4\pi\epsilon_0}\frac{kR}{r^2}\times 4\pi R^2$$

$$\boxed{\bar{E}_{out_{Surf}} = \frac{kR^3}{\epsilon_0 r^2}\hat{r}}$$

$$\bar{E}_{out_{volume}} = \frac{1}{4\pi\epsilon_0}\int\frac{\rho\, d\tau}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}(-3k)\times\frac{4}{3}\pi R^3$$

$$\boxed{\bar{E}_{out_{vol}} = -\frac{kR^3}{\epsilon_0 r^2}\hat{r}}$$

$$\boxed{\bar{E}_{out} = \bar{E}_{out_{Surf}} + \bar{E}_{out_{vol}} = 0}$$

$$E_{in} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int\rho\, d\tau = \frac{1}{4\pi\epsilon_0}\frac{-3k}{r^2}\times\frac{4}{3}\pi r^3 = \boxed{-\frac{kr}{\epsilon_0}\hat{r} = \bar{E}_{in}}$$

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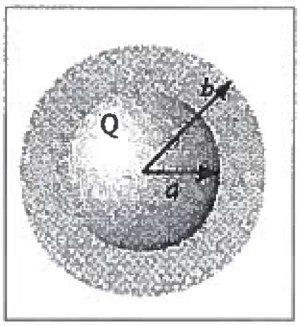
Q#5: (2+2+2+2+2)

A metal sphere of radius a carries a chare q. It is surrounded out to radius b by a linear dielectric material of permittivity ε.



(a) Find the displacement vector inside the sphere $0 < r < a$, in the dielectric medium $a < r < b$, and in the outer space $r > b$.
(b) Find the potential at the center of the sphere relative to infinity.
(c) Find the polarization $\vec{P}$.
(d) Find the volume bound charge ρ_b and surface bound charge σ_b.

a) $\bar{\nabla}\cdot\bar{D} = \rho_{free}$

$$\int_{vol} (\bar{\nabla}\cdot\bar{D})\, d\tau = \int \rho_{free}\, d\tau = Q$$

$$\int_{surf} \bar{D}\cdot d\bar{a} = Q \Rightarrow \boxed{\bar{D} = \frac{Q}{4\pi r^2}\hat{r}} \quad r > a$$

Electric field will be: $\boxed{\bar{D} = \epsilon \bar{E}}$

$$E = \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\hat{r} \quad b < r < a$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} \quad r > b$$

$$E = 0 \quad r < a$$

b) the Potential at the Centre of the sphere will be.

$$\Delta V = -\int_{\infty}^{0} \bar{E}\cdot d\bar{l} = -\int_{\infty}^{b} \bar{E}\cdot d\bar{r} - \int_{b}^{a} E\,dr - \int_{a}^{0} E\,dr$$

$$= -\frac{1}{4\pi\epsilon_0} Q\left[\frac{-1}{b} + \frac{1}{\infty}\right] - \frac{1}{4\pi\epsilon}\left[-\frac{1}{a} + \frac{1}{b}\right] - 0$$

$$\boxed{V(0) = \frac{Q}{4\pi}\left[\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right]}$$

Q#5 Continued

(c) $$\bar{P} = \epsilon_0 \chi_e \bar{E} = \frac{\epsilon_0 \chi_e Q}{4\pi \epsilon r^2} \hat{r} = \frac{\chi_e Q}{4\pi \epsilon_r r^2} \hat{r}$$

(d) bound volume charge

$$\rho_b = -\bar{\nabla}\cdot\bar{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\chi_e Q}{4\pi \epsilon_r r^2}\right) = 0$$

Surface bound charge

$$\sigma_b = \bar{P}\cdot\hat{n} = \begin{cases} \dfrac{\chi_e Q}{4\pi \epsilon_r b^2} & \text{at the outer surface} \\[2ex] \dfrac{-\chi_e Q}{4\pi \epsilon_r a^2} & \text{at the inner surface} \end{cases}$$