King Fahd University of Petroleum and Minerals – Physics Department

PHYS305 – Electricity and Magnetism I – Term 211 – Fall 2021

First Major exam 11 October, 2021 (Time 2:00 hours)

Q#1:

(a) Calculate the divergence of the following function:

$$\vec{v} = y^2\hat{\imath} + (2xy + z^2)\hat{\jmath} + 2yz\hat{k}$$

(b) Check the divergence theorem for this vector function using the cube in Figure 1 as your volume.

a) $\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x} y^2 + \frac{\partial}{\partial y}(2xy+z^2) + \frac{\partial}{\partial z}(2yz)$

$= 0 + 2x + 2y = 2x+2y$

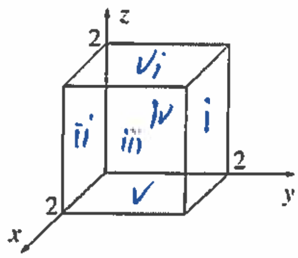

Figure 1

b) divergence theorem

$$\int (\bar{\nabla}\cdot\bar{v})\, d\tau = \oint \bar{v}\cdot d\bar{a}$$

$$L.H.S = \iiint (2x+2y)\,dx\,dy\,dz = \int_0^2\int_0^2\int_0^2 2x\,dx\,dy\,dz + \int_0^2\int_0^2\int_0^2 2y\,dx\,dy\,dz$$

$$= \frac{2x^2}{2}\, y\, z \Big|_0^2\Big|_0^2\Big|_0^2 + \frac{2y^2}{2}\, x\, z \Big|_0^2\Big|_0^2\Big|_0^2$$

$$= (2)^2 \times 2\times 2 + (2)^2\times 2\times 2 = 32$$

$$R.H.S = \oint \hat{v}\cdot d\bar{a} = \int_i \hat{v}\cdot d\bar{a} + \int_{ii} \hat{v}\cdot d\bar{a} + \int_{iii} \hat{v}\cdot d\bar{a} + \int_{iv} \vec{v}\, d\bar{a} + \int_{v} \hat{v}\cdot d\bar{a} + \int_{vi} \hat{v}\, d\bar{a}$$

$$= \int_{\substack{i\\ y=2}} \bar{v}\cdot dx\,dz\,\hat{j} + \int_{ii\ y=0} \hat{v}\cdot dx\,dz\,(-\hat{j}) + \int_{iii\ x=2} \hat{v}\cdot dy\,dz\,\hat{i} + \int_{iv\ x=0} \hat{v}\cdot dy\,dz\,(-\hat{i})$$

$$+ \int_{v\ z=0} \hat{v}\cdot dx\,dy\,(-\hat{k}) + \int_{vi\ z=2} \hat{v}\cdot dx\,dy\,\hat{k}$$

$\longrightarrow$

Q #1 (b) Continued

$$R.H.S = \underset{i\ y=2}{\int_0^2\int_0^2}(2xy+z^2)\,dx\,dz - \underset{ii\ y=0}{\int_0^2\int_0^2}(2xy+z^2)\,dx\,dz + \underset{iii\ x=2}{\int_0^2\int_0^2} y^2\,dy\,dz$$

$$- \underset{iv\ x=0}{\int_0^2\int_0^2} y^2\,dy\,dz - \underset{v\ z=0}{\int_0^2\int_0^2} 2yz\,dx\,dy + \underset{vi\ z=2}{\int_0^2\int_0^2} 2yz\,dx\,dy$$

$$R.H.S = \left(2\times\cancel{2}\times\frac{x^2}{\cancel{2}}\times z\Big|_0^2\Big|_0^2 + \frac{z^3}{3}\times x\Big|_0^2\Big|_0^2\right) - \left(2\times 0\times\frac{x^2}{2}\times z\Big|_0^2\Big|_0^2 + \frac{z^3}{3}\times x\Big|_0^2\Big|_0^2\right)$$

$$+ \frac{y^3}{3}\times z\Big|_0^2\Big|_0^2 - \frac{y^3}{3}\times z\Big|_0^2\Big|_0^2 - 2\times 0\times\frac{y^2}{2}\times x\Big|_0^2\Big|_0^2 + 2\times\cancel{2}\times\frac{y^2}{\cancel{2}}\times x\Big|_0^2\Big|_0^2$$

$$= 2\times(2)^2\times 2 + \frac{(2)^3}{3}\times 2 - 0 - \frac{2^3}{3}\times 2 + \frac{(2)^3}{3}\times 2 - \frac{(2)^3}{3}\times 2 - 0 + 2\times(2)^2\times$$

$$= 16 + \frac{16}{3} - \frac{16}{3} + \frac{16}{3} - \frac{16}{3} + 16 = 32$$

$$R.H.S = L.H.S$$

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Q#2: (a) Calculate the curl of the following vector function:

$$\vec{v} = xy\hat{\imath} + 2yz\,\hat{\jmath} + 3zx\,\hat{k}$$

(b) Check Stoke's theorem for this function for the square shown in Figure 2.

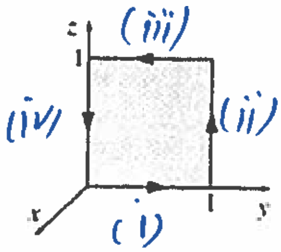

Figure 2

a) $$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3zx) - \frac{\partial}{\partial z}(2yz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(3zx) - \frac{\partial}{\partial z}(xy)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(xy)\right]$$

$$= \hat{i}\left[0 - 2y\right] - \hat{j}\left[3z - 0\right] + \hat{k}\left[0 - x\right]$$

$$\vec{\nabla}\times\vec{v} = -2y\,\hat{i} - 3z\,\hat{j} - x\,\hat{k}$$

b) Stoke's theorem

$$\int(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint\vec{v}\cdot d\vec{l}$$

$$L.H.S = \int(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \int_0^1\int_0^1 (-2y\,\hat{i} - 3z\,\hat{j} - x\,\hat{k})\cdot dy\,dz\,\hat{i}\Big|_{x=0}$$

$\longrightarrow$

Q#2(b) Continued

$$L.H.S = \int_0^1\int_0^1 -2y\, dy\, dz \Big|_{x=0} = -\frac{\cancel{2}y^2}{\cancel{2}} \times z \Big|_0^1 \Big|_0^1$$

$$= -(1)^2 \times 1 - 0 = -1$$

$$R.H.S. = \oint \bar{v} \cdot \bar{dl} = \int_{\substack{(i)\\ x=0,\, z=0}} \bar{v}\cdot\bar{dl} + \int_{\substack{(ii)\\ x=0,\, y=1}} \bar{v}\cdot\bar{dl} + \int_{\substack{(iii)\\ x=0\\ z=1}} \bar{v}\cdot\bar{dl}$$

$$+ \int_{\substack{(iv)\\ x=0,\, y=0}} \bar{v}\cdot\bar{dl}$$

$$= \int_{\substack{0\\ (i)\\ x=0,\, z=0}}^1 2yz\, dy + \int_{\substack{0\\ (ii)\\ x=0,\, y=1}}^1 3zx\, dz + \int_{\substack{1\\ (iii)\\ x=0,\, z=1}}^0 2yz\, dy + \int_{\substack{1\\ (iv)\\ x=0\\ y=0}}^0 3zx\, dz$$

$$= \frac{\cancel{2}xy^2}{\cancel{2}}\Big|_{\substack{0\\ x=0\\ z=0}}^1 + 3\times\frac{z^2}{2}x\Big|_{\substack{0\\ x=0,\, y=1}}^1 + \frac{\cancel{2}y^2}{\cancel{2}}\times z\Big|_{\substack{1\\ x=0,\, z=1}}^0 + 3\frac{z^2}{2}x\Big|_{\substack{1\\ x=0,\, y=0}}^0$$

$$= 0 + 0 + (0-(1)^2)\times 1 - 0 = -1$$

$$L.H.S = R.H.S$$

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Q#3:

(a) Evaluate the following integral: $\int_{\mathcal{V}} e^{-r}\left(\vec{\nabla}.\frac{\hat{r}}{r^2}\right)d\tau$; where $\mathcal{V}$ is a sphere of radius R centered at the origin.

(b) Prove that following function is not a possible electrostatic field:

$$\vec{E} = 2xy\hat{\imath} + 4yz\hat{\jmath} + 6xz\hat{k}$$

a) $$\int \bar{e}^{-r}\left(\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)\right)d\tau = \int_0^R\int_0^\pi\int_0^{2\pi} \bar{e}^{-r}\, 4\pi\,\delta^3(r)\, d\tau$$

$$= 4\pi\, e^{-0} = 4\pi$$

(b) $$\bar{\nabla}\times\bar{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & 4yz & 6xz \end{vmatrix}$$

$$= \hat{i}(0-4y) - \hat{j}(6z-0) + \hat{k}(0-2x)$$

$$\bar{\nabla}\times\bar{E} = -4y\hat{i} - 6z\hat{j} - 2x\hat{k}$$

Since $\bar{\nabla}\times\bar{E}$ is not equal to zero, so this vector is not an electrostatic field.

and for $z=0$

$$E_z = \frac{1}{2\varepsilon_0} \frac{\lambda r z}{(r^2+z^2)^{3/2}} = 0$$

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Q#4: Find electric field a distance z above a circular loop of radius r having a uniform linear charge density λ as shown in Figure 3. Check your answer for $z \gg r$ and at the center of the loop in the plane of the loop.

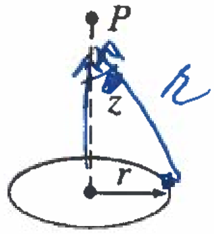

Figure 3

E-field due to this linear charge distribution is given by:

$$E = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}^2} = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\, dl}{\mathscr{r}^2}$$

$$\bar{r} + \bar{\mathscr{r}} = \bar{z} \Rightarrow \bar{\mathscr{r}} = \bar{z} - \bar{r}$$

$$\bar{\mathscr{r}}\cdot\bar{\mathscr{r}} = \mathscr{r}^2 = (\bar{z}-\bar{r})\cdot(\bar{z}-\bar{r}) = z^2 + r^2 - 2rz\cos\theta$$

$\theta = 90$

$$\mathscr{r}^2 = z^2 + r^2$$

$$E = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\lambda \times r\, d\theta}{(r^2+z^2)} = \frac{1}{4\pi\epsilon_0} \times \frac{\lambda \times r \times 2\pi}{(r^2+z^2)}$$

$$E = \frac{1}{2\epsilon_0}\frac{\lambda r}{(r^2+z^2)}$$

$$E_z = \frac{1}{2\epsilon_0}\frac{\lambda r}{(r^2+z^2)}\cos\theta$$

$$z = \mathscr{r}\cos\theta \Rightarrow \cos\theta = \frac{z}{\mathscr{r}} = \frac{z}{(r^2+z^2)^{1/2}}$$

$$E_z = \frac{1}{2\epsilon_0}\frac{\lambda r z}{(r^2+z^2)^{3/2}}$$

for $z \gg r$

$$E = \frac{1}{2\epsilon_0}\frac{\lambda r z}{z^3\left(1+\frac{r^2}{z^2}\right)^{3/2}} \simeq \frac{1}{2\epsilon_0}\frac{\lambda r}{z^2}$$

$$E = \frac{1}{4\pi\epsilon_0} \times \frac{\lambda \times 2\pi r}{z^2} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{z^2}} \quad z \gg R$$

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Q#5: Find the electric potential due to a uniformly charged spherical shell of radius R as shown in Figure 4. Check potentials for $z > R$ and $z < R$. Take potential at infinity to be zero.

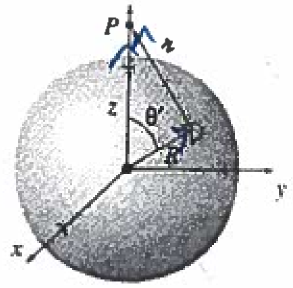

Figure 4

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma\, da}{r}$$

$$\bar{z} = \bar{R} + \bar{r}$$

$$\bar{r} = \bar{z} - \bar{R}$$

$$r^2 = (\bar{z}-\bar{R})\cdot(\bar{z}-\bar{R}) = z^2 + R^2 - 2Rz\cos\theta$$

$$V = \frac{1}{4\pi\epsilon_0} \times \sigma \times \int_0^{\pi}\int_0^{2\pi} \frac{R^2 \sin\theta\, d\theta\, d\phi}{(R^2+z^2-2Rz\cos\theta)^{1/2}}$$

$$= \frac{1}{\cancel{4}_2\pi\epsilon_0} \times \sigma \times \cancel{2}\pi R^2 \int_0^{\pi} \frac{\sin\theta\, d\theta}{(R^2+z^2-2Rz\cos\theta)^{1/2}}$$

$$\cos\theta = y \qquad -\sin\theta\, d\theta = dy$$

$$\theta = 0, \; y = 1, \quad \theta = \pi, \; y = -1$$

$$V = \frac{\sigma R^2}{2\epsilon_0}\int_1^{-1} \frac{-dy}{(R^2+z^2-2Rzy)^{1/2}} = \frac{-\sigma R^2}{\cancel{2}\epsilon_0} \times \cancel{2}(R^2+z^2-2Rzy)^{1/2}\Big|_{-1}^{+1}$$

$$= \frac{-\sigma R^2}{2\epsilon_0}\left[(R^2+z^2-2Rz)^{1/2} - (R^2+z^2+2Rz)^{1/2}\right] \times$$

Q#5 Continued

$$V = \frac{-\sigma R^2}{\cancel{2}Rz\epsilon_0}\left[(z-R)-(z+R)\right] \quad \text{for } z > R$$

$$V = \frac{-\sigma R^2}{\cancel{2}Rz\epsilon_0} \times -\cancel{2}R = \frac{\sigma R^2}{\epsilon_0 z}$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{\sigma \times 4\pi R^2}{z} = \frac{1}{4\pi\epsilon_0}\frac{q}{z}$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{q}{z}}$$ for $z > R$ outside the sphere.

for $z < R$

$$V = \frac{-\sigma R^2}{2Rz\epsilon_0} \times \left[(R-z)-(R+z)\right] \quad z < R$$

$$= \frac{+\sigma R^{\cancel{2}}}{\cancel{2}\cancel{R}\cancel{z}\epsilon_0} \times +\cancel{2}\cancel{z} = \frac{\sigma R^2}{R\epsilon_0} = \frac{1}{4\pi\epsilon_0}\frac{\sigma 4\pi R^2}{R}$$

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{q}{R}}$$ $z < R$ inside the sphere

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Q#6: Find the energy of a uniformly charged spherical shell of total charge q and radius R.

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau$$

$$= \frac{\epsilon_0}{2}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2 \times r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{q^2 \times 4\pi}{32\pi^2\epsilon_0}\left(\int_0^R 0 + \int_R^{\infty}\frac{1}{r^2}dr\right) = \frac{q^2}{8\pi\epsilon_0}\left(0 - \frac{1}{r}\Big|_R^{\infty}\right) = \frac{q^2}{8\pi\epsilon_0}\left(0 - \frac{1}{\infty} + \frac{1}{R}\right)$$

$$\boxed{W = \frac{q^2}{8\pi\epsilon_0 R}}$$

$$\boxed{W = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}}$$