1.2.3 The Operator ∇:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\right)\hat{\imath} + \left(\frac{\partial}{\partial y}\right)\hat{\jmath} + \left(\frac{\partial}{\partial z}\right)\hat{k}$$

Is not a vector by itself this is just an operator when it applies to a scalar or vector quantity, so del is a vector operator that acts on T, not a vector that multiplies with T.

But like an ordinary vector, we can either operate it on a scalar function or take a dot or cross product with another vector.

Gradient: $\vec{\nabla}\,(\mathrm{T})$

Divergence: $\vec{\nabla}.\vec{\mathrm{A}}$

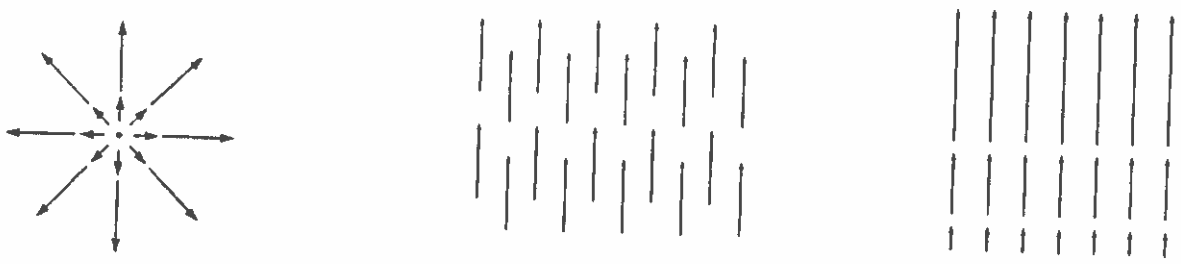
Curl: $\vec{\nabla} \times \vec{\mathrm{A}}$

1.2.4 The Divergence:

$$\vec{\nabla}.\vec{\mathrm{v}} = \left[\left(\frac{\partial}{\partial x}\right)\hat{\imath} + \left(\frac{\partial}{\partial y}\right)\hat{\jmath} + \left(\frac{\partial}{\partial z}\right)\hat{k}\right].\left[v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}\right]$$

$$\vec{\nabla}.\vec{\mathrm{v}} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Divergence of a vector is a scalar quantity. The divergence means how much a function spreads out from a certain point. If the function does not spread out at all (I,e it is uniform) then the divergence would be zero.



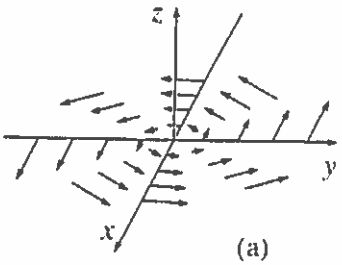
A point of positive divergence is a point where function spread out (like a source or faucet) and a point of negative divergence is a point where function collects together (like a sink or drain).

- **Example 6:** Suppose the functions in the above figures are written algebraically as $\vec{\mathrm{v}}_a = \vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ and $\vec{\mathrm{v}}_b = \hat{k}$ and $\vec{\mathrm{v}}_c = z\hat{k}$. Find the divergence of each function.
  Ans: (a) 3, (b) 0, (c) 1

1.2.4 The Curl:

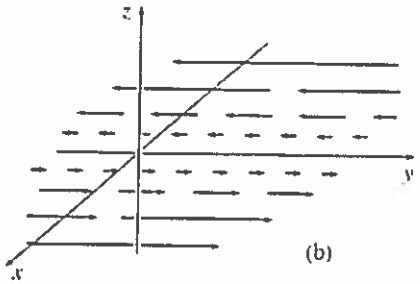
$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial y}\right)\hat{\imath} - \left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

$\vec{\nabla} \times \vec{v}$ is a measure of how much a vector curls around a point.



- **Example 7:** Suppose the functions in the above figures are written algebraically as $\vec{v}_a = -y\hat{\imath} + x\hat{\jmath}$ and $\vec{v}_b = x\hat{\jmath}$. Find the curl of these functions.

Ans: (a) $2\hat{k}$, (b) $\hat{k}$



1.2.6 Product Rules

The calculation of ordinary derivatives is facilitated by a number of general rules, such as the sum rule:

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx},$$

the rule for multiplying by a constant:

$$\frac{d}{dx}(kf) = k\frac{df}{dx},$$

the product rule:

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx},$$

and the quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\dfrac{df}{dx} - f\dfrac{dg}{dx}}{g^2}.$$

Similar relations hold for the vector derivatives. Thus,

$$\nabla(f+g) = \nabla f + \nabla g, \qquad \nabla\cdot(\mathbf{A}+\mathbf{B}) = (\nabla\cdot\mathbf{A}) + (\nabla\cdot\mathbf{B}),$$

$$\nabla\times(\mathbf{A}+\mathbf{B}) = (\nabla\times\mathbf{A}) + (\nabla\times\mathbf{B}),$$

and

$$\nabla(kf) = k\nabla f, \qquad \nabla\cdot(k\mathbf{A}) = k(\nabla\cdot\mathbf{A}), \qquad \nabla\times(k\mathbf{A}) = k(\nabla\times\mathbf{A}),$$

as you can check for yourself. The product rules are not quite so simple. There are two ways to construct a scalar as the product of two functions:

$$\begin{array}{ll} fg & \text{(product of two scalar functions),} \\ \mathbf{A}\cdot\mathbf{B} & \text{(dot product of two vector functions),} \end{array}$$

and two ways to make a vector:

$$\begin{array}{ll} f\mathbf{A} & \text{(scalar times vector),} \\ \mathbf{A}\times\mathbf{B} & \text{(cross product of two vectors).} \end{array}$$

Accordingly, there are *six* product rules, two for gradients:

(i) $$\nabla(fg) = f\nabla g + g\nabla f,$$

(ii) $$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A},$$

two for divergences:

(iii) $$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f),$$

(iv) $$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}),$$

and two for curls:

(v) $$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f),$$

(vi) $$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}).$$

1.2.7 Second Derivatives

By applying $\vec{\nabla}$ twice we can construct five species of second derivative.

The gradient $\vec{\nabla}T$ is a vector so we can take its divergence and curl.

(i) divergence $(\vec{\nabla}.\vec{\nabla}T)$ and
(ii) curl $(\vec{\nabla}\times\vec{\nabla}T)$.

Divergence is a scalar quantity, we can only take its gradient.

(iii) $\vec{\nabla}(\vec{\nabla}.\vec{v})$

The curl is a vector so we can take its divergence and curl.

(iv) $\vec{\nabla}.(\vec{\nabla}\times\vec{v})$
(v) $\vec{\nabla}\times(\vec{\nabla}\times\vec{v})$

(i) $$\vec{\nabla}.\vec{\nabla}T = \left(\left(\frac{\partial}{\partial x}\right)\hat{\imath} + \left(\frac{\partial}{\partial y}\right)\hat{\jmath} + \left(\frac{\partial}{\partial z}\right)\hat{k}\right).\left(\left(\frac{\partial T}{\partial x}\right)\hat{\imath} + \left(\frac{\partial T}{\partial y}\right)\hat{\jmath} + \left(\frac{\partial T}{\partial z}\right)\hat{k}\right)$$

$$\vec{\nabla}.\vec{\nabla}T = \left(\frac{\partial^2 T}{\partial x^2}\right) + \left(\frac{\partial^2 T}{\partial y^2}\right) + \left(\frac{\partial^2 T}{\partial z^2}\right) = \nabla^2 T$$

$\nabla^2 T$ is called the ***Laplacian*** of T.

- *Laplacian of a scalar T is a scalar quantity.*
- *Laplacian of a vector is a vector quantity.*

$$\nabla^2\vec{v} = \left(\frac{\partial^2 v_x}{\partial x^2}\right)\hat{\imath} + \left(\frac{\partial^2 v_y}{\partial y^2}\right)\hat{\jmath} + \left(\frac{\partial^2 v_z}{\partial z^2}\right)\hat{k}$$

- *The curl of a Laplacian is always zero:*

$$\vec{\nabla} \times (\vec{\nabla}T) = 0$$

Important to note that $\vec{\nabla} \times (\vec{\nabla}T)$ is zero not because $(\vec{\nabla} \times \vec{\nabla})T$ and $\vec{\nabla} \times \vec{\nabla}$ should be zero because cross product of a vector with itself is always zero.

NO, because $\vec{\nabla}$ is not a vector, it is an operator. We need to expand this expression and use:

$$\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial x}\right)$$

✓ $\vec{\nabla}(\vec{\nabla}.\vec{v})$ does not occur much in physical applications and has not specific name, this is just gradient of a divergence. $\vec{\nabla}(\vec{\nabla}.\vec{v})$ is not a Laplacian of $\vec{v}$.

$$\vec{\nabla}(\vec{\nabla}.\vec{v}) \neq (\vec{\nabla}.\vec{\nabla})\vec{v} = \nabla^2\vec{v}$$

- *The divergence of a curl is always zero.*

$$\vec{\nabla}.(\vec{\nabla} \times \vec{v}) = 0$$

- *The curl of a curl of a vector does not give any thing new:*

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla}.\vec{v}) - \nabla^2\vec{v}$$

The first term is just number 3 and the second term is Laplacian of a vector.

So there are only two kinds of second derivative **Laplacian** and the gradient-of-divergence which is not used often.

✓ **Example 8:** Calculate the Laplacian of $\vec{v} = x^2\hat{\imath} + 3xz^2\hat{\jmath} - 2xz\hat{k}$

Ans: $2\hat{\imath} + 6x\hat{\jmath}$

1.3 Integral Calculus

1.3.1 Line, Surface and Volume Integral

A line integral is the expression of the form:

$$\int_{a\,P}^{b} \vec{v}.\overrightarrow{dl}$$

Here $\vec{v}$ is a vector and $\overrightarrow{dl}$ is an infinitesimal displacement and integral is taken over a prescribed path P from point a to point b.

If the path forms a closed loop i.e. a=b then

$$\oint \vec{v}.\overrightarrow{dl}$$

An example of line integral is work, where

$$W = \int_a^b \vec{F}.\overrightarrow{dl}$$

Normally, the value of line integral depends on the path taken but in some cases it is independent of the path and only depends on the initial and final points on the line. The force that has this property is called **Conservative force.**