1.3.3 The Fundamental Theorem of Curl or Stoke's Theorem

$$\int_{a}^{b} (\vec{\nabla} \times \vec{v}).\overrightarrow{da} = \oint \vec{v}.\overrightarrow{dl}$$

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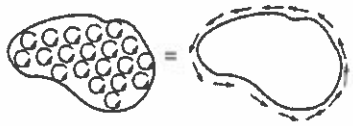
Integral of a derivative (curl) over a region is equal to the value of the function at the boundary. Here the boundary term is itself a closed line integral

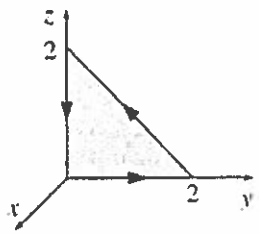
Integral of the curl over some surface (or flux of the curl through that surface) represents the total amount of swirl, and we can also determine this by going around the edge and find the flow at the boundary. Correct direction of area and line integral can be achieved by the right hand rule.



- $\int(\vec{\nabla} \times \vec{v}).\overrightarrow{da} = 0$ depends only on the boundary line not on the particular surface used.
- $\oint(\vec{\nabla} \times \vec{v}).\overrightarrow{da} = 0$, since the boundary line like the mouth of a balloon, shrinks down to a point so right side $\oint \vec{v}.\overrightarrow{dl} = 0$

Example 14: Test Stoke's theorem for the function $\vec{v} = xy\hat{\imath} + 2yz\hat{\jmath} + 3zx\hat{k}$, using the triangular shaded area in the figure below.

$z = 2 - y$



$$\bar{\nabla} \times \bar{v} = \begin{vmatrix} i & j & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & 2yz & 3zx \end{vmatrix}$$

$$= \hat{i}(0 - 2y) - \hat{j}(3z - 0) + \hat{k}(0 - x) = -2y\hat{i} - 3z\hat{j} - x\hat{k}$$

$$\bar{\nabla} \times \bar{v} \cdot d\bar{a} = (-2y\hat{i} - 3z\hat{j} - x\hat{k}) \cdot dy\, dz\, \hat{i}$$

$$= \cancel{-2y\,dx - 3z\,dy - x\,dz} \quad -2y\,dy\,dz$$

$$\int_S \bar{\nabla} \times \bar{v} \cdot d\bar{a} = \int_0^2 \int_0^{2-y} -2y\,dy\,dz = \int_0^2 -2y(2-y)\,dy$$

$$= -2\left(\frac{y^2}{2} + \frac{2y^3}{3}\right)\Big|_0^2$$

$$= -2 \times 4 \cdot 2 \cdot 8 = -8 + \frac{16}{3} \left(-\frac{8}{3}\right)$$

1.3.6 Integrations by parts

Using the derivative identity:

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

Integrating both sides:

$$\int_a^b \frac{d}{dx}(fg)\,dx = fg|_a^b = \int_a^b f\frac{dg}{dx} + \int_a^b g\frac{df}{dx}\,dx$$

$$\int_a^b f\frac{dg}{dx} = fg|_a^b - \int_a^b g\frac{df}{dx}\,dx$$

Example 15: Evaluate the integral

$$\int_0^\infty xe^{-x}\,dx$$

$f = x, \quad g = -e^{-x} \Rightarrow \frac{dg}{dx} = +e^{-x}$

$$\int_0^\infty f\frac{dg}{dx}\,dx = f\times g\Big|_0^\infty - \int_0^\infty g\frac{df}{dx}\,dx$$

$$\int_0^\infty x e^{-x}\,dx = -xe^{-x}\Big|_0^\infty - \int_0^\infty -e^{-x}\times 1\,dx$$

$$= 0 + \frac{e^{-x}}{-1}\Big|_0^\infty = -1\left(\frac{1}{e^\infty} - \frac{1}{e^0}\right) = 1$$