1.3.3 The Fundamental Theorem of Divergence

The fundamental theorem of divergence states that:

$$\int_{a}^{b} (\vec{\nabla}.\vec{v})d\tau = \oint \vec{v}.\overrightarrow{da}$$

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This theorem is a special theorem and is called either **Gauss's theorem**, **Green's theorem** or just **divergence theorem**.

This theorem also states that integral of a derivative is equal to the value of the function at its boundaries. Since integral is volume integral and hence boundary of a volume is a closed surface.

If v represents a flow of incompressible fluid then flux of a liquid is equal to the total amount of fluid flow out of the boundaries.

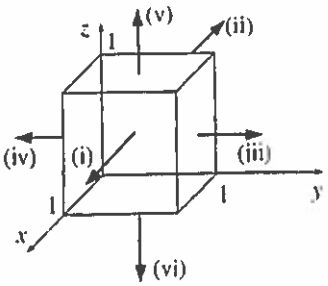
Divergence means fluid spreading out (like a faucet). So if we have lots of faucets pouring out incompressible fluid an equal amount of fluid will be pouring out of the boundaries of the region.

(i) Either we can count all faucets and add the output of all.
(ii) Or go to the boundary and measure the flow at each point and add all up.

$$\int (faucets\ within\ the\ volume) = \oint (flow\ out\ through\ the\ surface)$$

- **Example 13:** Check the divergence theorem using the function $\vec{v} = y^2\hat{\imath} + (2xy + z^2)\hat{\jmath} + 2yz\hat{k}$ and the unit cube situated at the origin.

Ans: 2



$$\vec{\nabla}.\vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(y^2\hat{i} + (2xy + z^2)\hat{j} + 2yz\hat{k}\right)$$

$$= \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}2yz$$

$$= 0 + (2x + 0) + 2y = 2x + 2y$$

$$\int (\vec{\nabla}.\vec{v})\,d\tau = \int_0^1\int_0^1\int_0^1 (2x + 2y)\,dx\,dy\,dz = \frac{2x^2}{2}\Big|_0^1 + \frac{2y^2}{2}\Big|_0^1$$

$$= 1 + 1 = 2$$