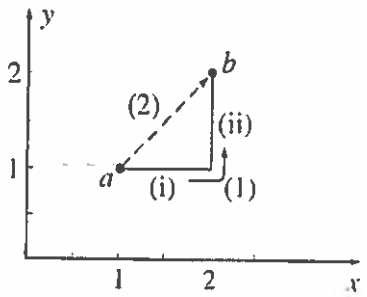
➢ **Example 9:** Calculate the line integral of the function $\vec{v} = y^2\hat{\imath} + 2x(y+1)\hat{\jmath}$ from the point a(1,1,0) to a point b(2,2,0) along the paths 1 and 2 in the figure.
What is $\oint \vec{v}.\overrightarrow{dl}$ for the loop that goes from a to b along path (1) and come back to a along path (2).



Ans: (i) 11, (ii) 10 (iii) 1

$$\int_a^b \vec{v}.d\vec{l} = \int \left(y^2\hat{i} + 2x(y+1)\hat{j}\right).(dx\hat{i} + dy\hat{j})$$

$$= \int_{y=1}^{2} y^2 dx + \int_{x=2}^{2} 2x(y+1)\,dy$$

$$= 1\times x\Big|_1^2 + 2\times 2\left(\frac{y^2}{2} + y\right)\Big|_1^2 = 1\times(2-1) + 4\left(\frac{4}{2} + 2 - \left(\frac{1}{2}+1\right)\right)$$

$$= 1 + 4\left(2+2-\frac{3}{2}\right) = 1 + \frac{20}{2} = 11$$

Path 2

$$\int_{y=x}^{2} y^2 dx + \int 2x(y+1)\,dy = \int_1^2 x^2 dx + \int_1^2 (2y^2 + 2y)\,dy$$

$$= \frac{x^3}{3}\Big|_1^2 + 2\times\left(\frac{y^3}{3} + \frac{y^2}{2}\right)\Big|_1^2 = \frac{1}{3}[8-1] + 2\left[\left(\frac{8}{3}+\frac{4}{2}\right) - \left(\frac{1}{3}+\frac{1}{2}\right)\right]$$

$$= \frac{7}{3} + 2\left(\frac{28}{6} - \frac{5}{6}\right) = \frac{20}{3}$$

$$= 10$$

1.3.1 Line, Surface and Volume Integral

A surface integral is the expression of the form:

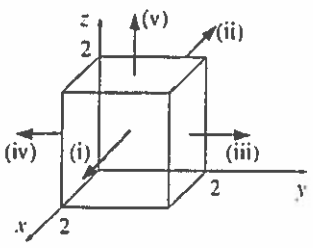
$$\int_S \vec{v}.\overrightarrow{da}$$

Where $\vec{v}$ is a vector and $\overrightarrow{da}$ is an infinitesimal area, with direction perpendicular to the surface. If a surface is closed like a balloon then integral becomes:

$$\oint \vec{v}.\overrightarrow{da}$$

Outward direction can be taken as positive direction of area for a closed surface but for the open surface, it is arbitrary.

➢ **Example 10:** Calculate the surface integral of $\vec{v} = 2xz\hat{\imath} + (x+2)\hat{\jmath} + y(z^2-3)\hat{k}$ over five sides excluding the bottom of the cubical box in the figure. Let upward and outward be the positive direction as indicated by the arrows.



Ans: 20

1.3.1 Volume Integral

A volume integral is the expression of the form:

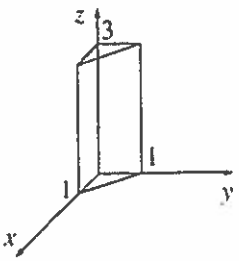
$$\int_{V} T\, d\tau$$

Where T is a scalar and $d\tau = dx\, dy\, dz$ is an infinitesimal volume element. If T is a density of a material then the volume integral will give total mass.

$$\int \vec{v}\, d\tau = \int (v_x\hat{\imath} + v_x\hat{\imath} + v_x\hat{\imath})d\tau = \hat{\imath}\int v_x d\tau + \hat{\jmath}\int v_y d\tau + \hat{k}\int v_z d\tau$$

- **Example 11:** Calculate the volume integral of $T = xyz^2$ over the prism in the figure.

  Ans: 3/8



$y \to 0 \to 1$

$z \to 0 \to 3$

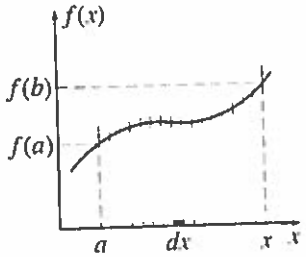
$x \to 0 \to 1-y$

$$\int T\, d\tau = \int_0^1\int_0^{1-y}\int_0^3 (xyz^2)\, dx\, dy\, dz$$

1.3.2 The Fundamental Theorem of Calculus

Suppose f(x) is a function of one variable, the fundamental theorem of calculus states that:

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$



Or

$$\int_a^b F(x)\,dx = f(b) - f(a)$$

$f(x) = \frac{x^2}{2}$

Where $\frac{df}{dx} = F(x)$, so this theorem helps us to integrate F(x),

$F(x) = x$

we need to figure out a function f(x) whose derivative is F.

The fundamental theorem states that if we chop-up the interval from a to b into tiny pieces, dx, and add up increment df from all those pieces then the result is equal to the total change $f(b) - f(a)$.

In other words, **"integral of a derivative over an interval is given by the value of the function at the end points"**.

1.3.3 The Fundamental Theorem of Gradients

$$\int_a^b (\vec{\nabla} T).\vec{dl} = T(b) - T(a)$$

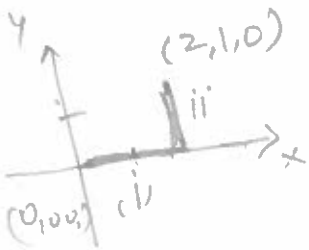
Suppose we want to measure the height of a building, we can climb the stairs and use the ruler to measure the height at each step and then add all of those heights, or we know the coordinate at the top and at the bottom and subtract the two, we will get the same answer.

- ✓ The line integral of the Gradient ($\int_a^b (\vec{\nabla} T).dl$) are path independent; they only depend on the initial and final positions.
- ✓ $\oint (\vec{\nabla} T).dl = 0$ because initial and final points are the same.

➢ **Example 12:** Let $T = xy^2$ and take a point a at the origin (0,0,0) and point b (2,1,0). Check the fundamental theorem of gradients.

Ans: 2

$$\int_a^b \vec{\nabla} T.(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = \int_0^2 \frac{\partial T}{\partial x}dx + \int_0^1 \frac{\partial T}{\partial y}dy + \int_0^0 \frac{\partial T}{\partial z}dz$$



$$= \int_0^2 y^2 dx + \int_0^1 2xy\,dy + 0$$

$$= \frac{y^2 x}{1}\Big|_0^2 + \frac{2x\,y^2}{x}\Big|_0^1 = 0 + 2\times(1)^2 = 2$$

$$T(b) - T(0) = 2\times(1)^2 - 0 = 2$$