Chapter 3. Special Techniques for Calculating Potentials

Given a stationary charge distribution $\rho(r)$ we can, in principle, calculate the electric field:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(r')}{(\Delta r)^2} \Delta\hat{r} \; d\tau'$$

Where $\Delta\vec{r} = \vec{r}' - \vec{r}$. This integral involves a vector as an integrand and is, in general, difficult to calculate. In most cases it is easier to evaluate first the electrostatic potential V which is defined as

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o} \int \frac{\rho(r')}{\Delta r} \; d\tau'$$

since the integrand of the integral is a scalar. The corresponding electric field $\vec{E}$ can then be obtained from the gradient of V since

$$\vec{E} = -\vec{\nabla}V$$

The electrostatic potential V can only be evaluated analytically for the simplest charge configurations. In addition, in many electrostatic problems, conductors are involved and the charge distribution ρ is not known in advance (only the total charge on each conductor is known).
A better approach to determine the electrostatic potential is to start with **Poisson's equation**

$$\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_o}$$

Very often we only want to determine the potential in a region where $\rho = 0$. In this region Poisson's equation reduces to **Laplace's equation**

$$\vec{\nabla}^2 V = 0$$

There are an infinite number of functions that satisfy Laplace's equation and the appropriate solution is selected by specifying the appropriate **boundary conditions**.

This Chapter will concentrate on the various techniques that can be used to calculate the solutions of Laplace's equation and on the boundary conditions required to uniquely determine a solution.

3.1. Solutions of Laplace's Equation in One-, Two, and Three Dimensions

3.1.1. Laplace's Equation in One Dimension

In one dimension the electrostatic potential *V* depends on only one variable *x*. The electrostatic potential *V*(*x*) is a solution of the one-dimensional Laplace equation

$$\frac{d^2V}{dx^2} = 0$$

The general solution of this equation is

$$V(x) = \mathrm{mx} + \mathrm{b}$$

where *m* and *b* are arbitrary constants. These constants are fixed when the value of the potential is specified at two different positions.

Example 1:
Consider a one-dimensional world with two point conductors located at *x* = 0 m and at *x* = 10 m. The conductor at *x* = 0 m is grounded and the conductor at *x* = 10 m is kept at a constant potential of 200 V. Determine *V(x)*.

The boundary conditions for *V* are

$$V(0) = \mathrm{b} = 0\mathrm{V}$$

and

$V(10) = \mathrm{m} * 10 + 0 = 200\ \mathrm{V}$

$$\mathrm{m} = 20\ V/m$$

The first boundary condition shows that *b* = 0 V and the second boundary condition shows that m = 20 V/m. The electrostatic potential for this system of conductors is thus

$$V(x) = 20x$$

The corresponding electric field can be obtained from the gradient of *V*

$\mathrm{E(x)} = -\frac{\mathrm{dV}}{\mathrm{dx}} = -20\ V/m$

The boundary conditions used here, can be used to specify the electrostatic potential between *x* = 0 m and *x* = 10 m but not in the region *x* < 0 m and *x* > 10 m. If the solution obtained here was the general solution for all *x*, then *V* would approach ∞ when *x* approaches infinity and *V* would approach minus infinity when *x* approaches minus infinity.

The boundary conditions therefore provide the information necessary to uniquely define a solution to Laplace's equation, but they also define the boundary of the region where this solution is valid (in this example 0 m < *x* < 10 m).

The following properties are true for any solution of the one-dimensional Laplace equation:

Property 1:
$V(x)$ is the average of $V(x + R)$ and $V(x - R)$ for any R as long as $x + R$ and $x - R$ are located in the region between the boundary points. This property is easy to prove:

$$\frac{V(x+R)+V(x-R)}{2} = \frac{m(x+R)+b+m(x-R)+b}{2} = mx + b = V(x)$$

This property immediately suggests a powerful analytical method to determine the solution of Laplace's equation. If the boundary values of V are:

$$V(x = a) = V_a$$

and

$$V(x = b) = V_b$$

then property 1 can be used to determine the value of the potential at $(a + b)/2$:

$$V\left(x = \frac{a+b}{2}\right) = \frac{1}{2}[V_a + V_b]$$

Next we can determine the value of the potential at $x = (3a + b)/4$ and at $x = (a + 3b)/4$:

$$V\left(x = \frac{3a+b}{2}\right) = \frac{1}{2}\left[V(x = a) + V\left(x = \frac{a+b}{2}\right)\right] = \frac{1}{2}\left[\frac{3}{2}V_a + \frac{1}{2}V_b\right]$$
$$V\left(x = \frac{a+3b}{2}\right) = \frac{1}{2}\left[V(x = \frac{a+b}{2}) + V(x = b)\right] = \frac{1}{2}\left[\frac{1}{2}V_a + \frac{3}{2}V_b\right]$$

This process can be repeated and V can be calculated in this manner at any point between $x = a$ and $x = b$ (but not in the region $x > b$ and $x < a$).

Property 2:
The solution of Laplace's equation can not have local maxima or minima. Extreme values must occur at the end points (the boundaries). This is a direct consequence of property 1.

Property 2 has an important consequence: a charged particle can not be held in stable equilibrium by electrostatic forces alone (**Earnshaw's Theorem**). A particle is in a stable equilibrium if it is located at a position where the potential has a minimum value. A small displacement away from the equilibrium position will increase the electrostatic potential of the particle, and a restoring force will try to move the particle back to its equilibrium position. However, since there can be no local maxima or minima in the electrostatic potential, the particle cannot be held in stable equilibrium by just electrostatic forces.

3.1.2. Laplace's Equation in Two Dimensions

In two dimensions the electrostatic potential depends on two variables *x* and *y*. Laplace's equation now becomes

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

This equation does not have a simple analytical solution as the one-dimensional Laplace equation does. However, the properties of solutions of the one-dimensional Laplace equation are also valid for solutions of the two-dimensional Laplace equation:

Property 1:
The value of *V* at a point (*x*, *y*) is equal to the average value of *V* around this point

$$V(x,y)=\frac{1}{2\pi R}\oint_{\text{circle}} V\ R\ d\phi$$

where the path integral is along a circle of arbitrary radius, centered at (*x*, *y*) and with radius *R*.

Property 2:
V has no local maxima or minima; all extremes occur at the boundaries.

3.1.3. Laplace's Equation in Three Dimensions

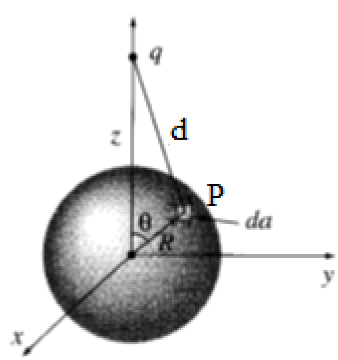
In three dimensions the electrostatic potential depends on three variables *x*, *y*, and *z*. Laplace's equation now becomes

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

This equation does not have a simple analytical solution as the one-dimensional Laplace equation does. However, the properties of solutions of the one-dimensional Laplace equation are also valid for solutions of the three-dimensional Laplace equation:

Property 1:
The value of *V* at a point (*x*, *y*, *z*) is equal to the average value of *V* around this point

$$V(x,y)=\frac{1}{4\pi R^2}\oint_{\text{sphere}} VR^2\sin\theta\ d\theta\ d\phi$$



where the surface integral is across the surface of a sphere of arbitrary radius, centered at (*x*,*y*,*z*) and with radius *R*.

To prove this property of V consider the electrostatic potential generated by a point charge q located on the z axis, a distance r away from the center of a sphere of radius R (see Figure 3.1). The potential at P, generated by charge q, is equal to

$$V_p = \frac{1}{4\pi\varepsilon_o}\frac{q}{d}$$

where d is the distance between q and surface patch. Using the cosine rule we can express d in terms of r, R and θ

$$d^2 = z^2 + R^2 - 2zR\cos\theta$$

The potential at P due to charge q is therefore equal to

$$V_p = \frac{1}{4\pi\varepsilon_o}\frac{q}{\sqrt{z^2 + R^2 - 2zR\cos\theta}}$$

The average potential on the surface of the sphere can be obtained by integrating V_p across the surface of the sphere. The average potential is equal to

$$V_{avg} = \frac{1}{4\pi R^2}\int V_p R^2 \sin\theta \, d\theta \, d\phi = \frac{1}{4\pi\varepsilon_o}\frac{q}{z}$$

which is equal to the potential due to q at the center of the sphere. Applying the principle of superposition it is easy to show that the average potential generated by a collection of point charges is equal to the net potential they produce at the center of the sphere.

Property 2:
The electrostatic potential V has no local maxima or minima; all extremes occur at the boundaries.

Example 3:
Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r. Then do the same for cylindrical coordinates.

Laplace's equation in spherical coordinates is given by

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

If V is only a function of r then $\frac{\partial V}{\partial\theta} = 0$ and $\frac{\partial V}{\partial\phi} = 0$

Therefore, Laplace's equation can be rewritten as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

The solution V of this second-order differential equation must satisfy the following first-order differential equation:

$$r^2 \frac{\partial V}{\partial r} = constant = a$$

This differential equation can be rewritten as

$$\frac{\partial V}{\partial r} = \frac{a}{r^2}$$

The general solution of this first-order differential equation is

$$V(r) = -\frac{a}{r} + b$$

where b is a constant. If $V = 0$ at infinity then b must be equal to zero, and consequently

$$V(r) = -\frac{a}{r}$$

Laplace's equation in cylindrical coordinates:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

If V is only a function of r then $\frac{\partial V}{\partial \phi} = 0$ and $\frac{\partial V}{\partial z} = 0$

Therefore, Laplace's equation can be rewritten as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

The solution V of this second-order differential equation must satisfy the following first-order differential equation:

$$r\frac{\partial V}{\partial r} = a = constant$$

This differential equation can be rewritten as

$\frac{\partial V}{\partial r} = \frac{a}{r}$

The general solution of this first-order differential equation is

$$V(r) = a\ln(r) + b$$

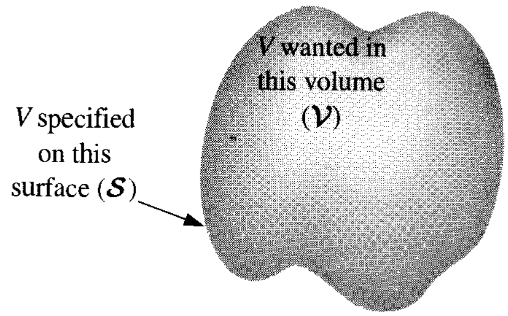
where b is a constant. The constants a and b are determined by the boundary conditions.

3.1.4. Uniqueness Theorems

Consider a volume within which the charge density is equal to zero. Suppose that the value of the electrostatic potential is specified at every point on the surface of this volume.

The **first uniqueness theorem** states that **in this case the solution of Laplace's equation is uniquely defined**.

To prove the first uniqueness theorem we will consider what happens when there are two solutions V_1 and V_2 of Laplace's equation in the volume shown in the figure. Since V_1 and V_2 are solutions of Laplace's equation, we know that



$\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$

Since both V_1 and V_2 are solutions, they must have the same value on the boundary. Thus $V_1 = V_2$ on the boundary of the volume.

Now consider a third function V_3, which is the difference between V_1 and V_2

$$V_3 = V_2 - V_1$$

The function V_3 is also a solution of Laplace's equation. This can be demonstrated easily:

$$\nabla^2 V_3 = \nabla^2 V_2 - \nabla^2 V_1 = 0$$

The value of the function V_3 is equal to zero on the boundary of the volume since $V_1 = V_2$ there. However, property 2 of any solution of Laplace's equation states that it can have no local maxima or minima and that the extreme values of the solution must occur at the boundaries. Since V_3 is a solution of Laplace's equation and its value is zero everywhere on the boundary of the volume, the maximum and minimum value of V_3 must be equal to zero. Therefore, V_3 must be equal to zero everywhere. This immediately implies that everywhere:

$$V_1 = V_2$$

This proves that there can be no two different functions V_1 and V_2 that are solutions of Laplace's equation and satisfy the same boundary conditions.

Therefore, the solution of Laplace's equation is uniquely determined if its value is a specified function on all boundaries of the region.

This also indicates that it does not matter how you come by your solution:

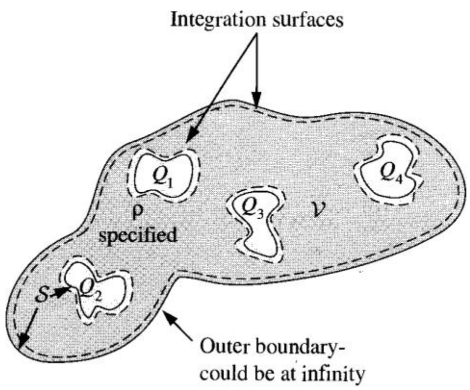
As long as (a) it is a solution of Laplace's equation, and (b) it has the correct value on the boundaries, then it is the right and only solution.

The first uniqueness theorem can only be applied in those regions that are free of charge and surrounded by a boundary with a known potential (not necessarily constant).

In the laboratory the boundaries are usually conductors connected to batteries to keep them at a fixed potential. In many other electrostatic problems, we do not know the potential at the boundaries of the system. Instead, we might know the total charge on the various conductors that make up the system (note: knowing the total charge on a conductor does not imply a knowledge of the charge distribution ρ since it is influenced by the presence of the other conductors).

In addition to the conductors that make up the system, there might be a charge distribution ρ filling the regions between the conductors. For this type of system, the first uniqueness theorem does not apply.

The second uniqueness theorem states that the electric field is uniquely determined if the total charge on each conductor is given and the charge distribution in the regions between the conductors is known.



The proof of the second uniqueness theorem is similar to the proof of the first uniqueness theorem. Suppose that there are two fields $\vec{E}_1$ and $\vec{E}_2$ that are solutions of Poisson's equation in the region between the conductors. Thus:

$\vec{\nabla}.\vec{E}_1 = \frac{\rho}{\varepsilon_o}$ and $\vec{\nabla}.\vec{E}_2 = \frac{\rho}{\varepsilon_o}$

where ρ is the charge density at the point where the electric field is evaluated.

The surface integrals of $\vec{E}_1$ and $\vec{E}_2$, evaluated using a surface that is just outside one of the conductors with charge Q_i,:

$$\int_{\substack{\text{Surface}\\ \text{conductor i}}} \vec{E}_1.d\vec{a} = \frac{Q_i}{\varepsilon_o} \ ; \ \int_{\substack{\text{Surface}\\ \text{conductor i}}} \vec{E}_2.d\vec{a} = \frac{Q_i}{\varepsilon_o}$$

The difference $\vec{E}_3 = \vec{E}_2 - \vec{E}_1$ satisfies the following equations:

$$\vec{\nabla}.\vec{E}_3 = \vec{\nabla}.\vec{E}_2 - \vec{\nabla}.\vec{E}_1 = \frac{\rho}{\varepsilon_o} - \frac{\rho}{\varepsilon_o} = 0$$

$$\int_{\substack{\text{Surface}\\ \text{conductor i}}} \vec{E}_3.d\vec{a} = \int_{\substack{\text{Surface}\\ \text{conductor i}}} \vec{E}_2.d\vec{a} - \int_{\substack{\text{Surface}\\ \text{conductor i}}} \vec{E}_1.d\vec{a} = \frac{Q_i}{\varepsilon_o} - \frac{Q_i}{\varepsilon_o} = 0$$

Consider the surface integral of $\vec{E}_3$, integrated over all surfaces (the surface of all conductors and the outer surface). Since the potential on the surface of any conductor is constant, the electrostatic potential associated with $\vec{E}_2$ and $\vec{E}_1$ must also be constant on the surface of each conductor.

Therefore, $V_3 = V_2 - V_1$ will also be constant on the surface of each conductor. The surface integral of $V_3\vec{E}_3$ over the surface of conductor *i* can be written as

$$\int_{\substack{\text{Surface}\\ \text{conductor i}}} V_3\vec{E}_3.d\vec{a} = V_3 \int_{\substack{\text{Surface}\\ \text{conductor i}}} \vec{E}_3.d\vec{a} = 0$$

Since the surface integral of $V_3\vec{E}_3$ over the surface of conductor *i* is equal to zero, the surface integral of $V_3\vec{E}_3$ over all conductor surfaces will also be equal to zero. The surface integral of $V_3\vec{E}_3$ over the outer surface will also be equal to zero since $V_3 = 0$ on this surface. Thus:

$$\int_{\text{All Surface}} V_3\vec{E}_3 . d\vec{a} = 0$$

Using product rule from chapter 1:

$$\vec{\nabla}.\left(V_3\vec{E}_3\right) = V_3\left(\vec{\nabla}.\vec{E}_3\right) + \vec{E}_3.\vec{\nabla}V_3$$

The surface integral of $V_3\vec{E}_3$ can be rewritten using Green's identity as

$$\int_{\text{All surface}} V_3\vec{E}_3 . d\vec{a} = \int_{\substack{\text{Volume b/w}\\ \text{conductors}}} V_3\left(\vec{\nabla}.\vec{E}_3\right)d\tau + \int_{\substack{\text{Volume b/w}\\ \text{conductors}}} \vec{E}_3.\vec{\nabla}V_3 \; d\tau$$

$$\int V_3\left(\vec{\nabla}.\vec{E}_3\right)d\tau + \int -\vec{E}_3.\vec{\mathrm{E}}_3 \; d\tau = 0$$

Since $\vec{\nabla}.\vec{E}_3 = 0$ from above:

$$\int_{\substack{\text{Volume b/w}\\ \text{conductors}}} E_3^2 \; d\tau = 0$$

where the volume integration is over all space between the conductors and the outer surface. Since E_3^2 is always positive, the volume integral of E_3^2 can only be equal to zero if:

$E_3^2 = 0$ everywhere.

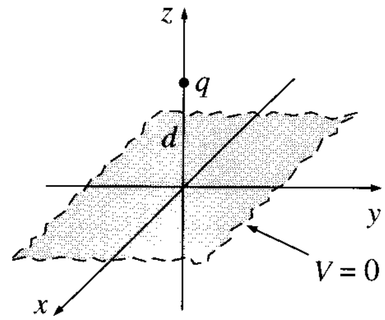
This implies immediately that $\vec{E}_2 = \vec{E}_1$ everywhere, and proves the second uniqueness theorem.

3.2. Method of Images

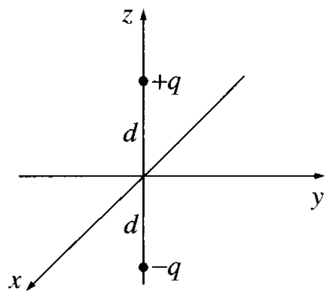
Consider a point charge q held at a distance d above an infinite grounded conducting plane as shown in the figure below. The electrostatic potential of this system must satisfy the following two boundary conditions:

$$V(x,y,0) = 0$$

$$V(x,y,z) \rightarrow 0 \; when \begin{cases} x \rightarrow \infty \\ y \rightarrow \infty \\ z \rightarrow \infty \end{cases}$$



A direct calculation of the electrostatic potential cannot be carried out since the charge distribution on the grounded conductor is unknown. **<u>Note</u>**: the charge distribution on the surface of a grounded conductor does not need to be zero.



Consider a second system, consisting of two point charges +*q* and -*q*, located at *z* = *d* and *z* = -*d*, respectively as shown in figure 2. The electrostatic potential generated by these two charges can be calculated directly at any point in space.

At a point *P* = (*x*, *y*, 0) on the *xy* plane the electrostatic potential is equal to

$$V(x,y,0) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{x^2+y^2+d^2}} + \frac{-q}{\sqrt{x^2+y^2+d^2}}\right] = 0$$

The potential of this system at infinity will approach zero since the potential generated by each charge will decrea
r .

Therefore, the electrostatic potential generated by the two charges satisfies the same boundary conditions as the original system in Fig.1.

Since the charge distribution in the region *z* > 0 (bounded by the *xy* plane boundary and the boundary at infinity) for the two systems is identical, the corollary of the **first uniqueness theorem** states that the **electrostatic potential in this region is uniquely defined**.

Therefore, if we find **any function** that satisfies the boundary conditions and Poisson's equation, it will be the right answer. Consider a point (*x*, *y*, *z*) with *z* > 0. The electrostatic potential at this point can be calculated easily for the charge distribution shown in Figure 3.5. It is equal to

$$V(x,y,0) = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Since this solution satisfies the boundary conditions, it must be the correct solution in the region ***z* > 0** for the system shown in Fig. 1.

This technique of using image charges to obtain the electrostatic potential in some region of space is called the **method of images**.

The electrostatic potential can be used to calculate the charge distribution on the grounded conductor. Since the electric field inside the conductor is equal to zero, the boundary condition for $\vec{E}$ shows that the electric field right outside the conductor is equal to

$$\vec{\boldsymbol{E}}_{\boldsymbol{outside}} = \frac{\boldsymbol{\sigma}}{\boldsymbol{\varepsilon_o}}\hat{\boldsymbol{n}}$$

where σ is the surface charge density and $\hat{n}$ is the unit vector normal to the surface of the conductor. Expressing the electric field in terms of the electrostatic potential V we can rewrite this equation as

$$\boldsymbol{\sigma = \varepsilon_o E_z = -\varepsilon_o \frac{\partial V}{\partial z}\bigg|_{z=0}}$$

Substituting the solution for V in this equation we find

$$\boldsymbol{\sigma} = -\frac{\boldsymbol{q}}{\boldsymbol{4\pi}}\left[\frac{-(z-d)}{(x^2+y^2+(z-d)^2)^{\frac{3}{2}}} + \frac{(z+d)}{(x^2+y^2+(z+d)^2)^{\frac{3}{2}}}\right]_{\boldsymbol{z=0}} = -\frac{\boldsymbol{q}}{\boldsymbol{2\pi}}\frac{\boldsymbol{d}}{\boldsymbol{(x^2+y^2+d^2)^{3/2}}}$$

The induced charge distribution is negative and the charge density is greatest at (x = 0, y = 0, z = 0). The total charge on the conductor can be calculated by surface integrating of σ:

$$Q_{total} = \int \sigma da = \int_0^{2\pi}\int_0^{\infty} \sigma(r) r\, dr\, d\theta$$

Where $r = \sqrt{x^2+y^2}$. By substituting the expression for σ in the integral we obtain

$$Q_{total} = -qd\int_0^{\infty} \frac{1}{(r^2+d^2)^{\frac{3}{2}}}\, rdr = qd\frac{1}{(r^2+d^2)^{\frac{1}{2}}}\bigg|_0^{\infty} = qd\left[0 - \frac{1}{d}\right] = -q$$

As a result of the induced surface charge on the conductor, the point charge q will be attracted towards the conductor.

Since the electrostatic potential generated by the charge and image-charge system is the same as the charge-conductor system in the region where $z > 0$, the associated electric field (and consequently the force on point charge q) will also be the same.

The force exerted on point charge q can be obtained immediately by calculating the force exerted on the point charge by the image charge. This force is equal to

$$\vec{F} = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{(2d)^2}\hat{k}$$

The total electrostatic energy of the charge and grounded conductor and charge image-charge system is not the same.

The electric field in the image-charge system is present everywhere, and the magnitude of the electric field at (*x*, *y*, *z*) will be the same as the magnitude of the electric field at (*x*, *y*, -*z*). On the other hand, in the real system the electric field will only be non-zero in the region with $z > 0$. Since the electrostatic energy of a system is proportional to the volume integral of $E^2$the electrostatic energy of the real system will be 1/2 of the electrostatic energy of the image-charge system (only 1/2 of the total volume has a non-zero electric field in the real system).

The electrostatic energy of the image-charge system is equal to

$$\boldsymbol{W_{image} = -q\Delta V = -q * \frac{1}{4\pi\varepsilon_o}\frac{q}{2d} = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{2d}}$$

The electrostatic energy of the real system is therefore equal to

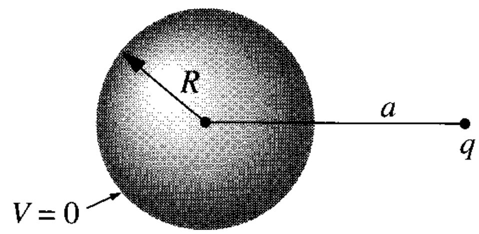
$$\boldsymbol{W = \frac{1}{2}W_{image} = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{4d}}$$

The electrostatic energy of the real system can also be obtained by calculating the work required to be done to assemble the system. In order to move the charge *q* to its final position we will have to exert a force opposite to the force exerted on it by the grounded conductor. The work done to move the charge from infinity along the *z* axis to *z* = *d* is equal to

$$\boldsymbol{W = \frac{1}{4\pi\varepsilon_o}\int_{\infty}^{d}\frac{q^2}{4z^2}dz = \frac{1}{4\pi\varepsilon_o}\frac{-q^2}{4z}\Bigg|_{\infty}^{d} = -\frac{1}{4\pi\varepsilon_o}\frac{q^2}{4d}}$$

which is identical to the result obtained using the electrostatic potential energy of the image-charge system.

Example 4:
A point charge q is situated a distance s from the center of a grounded conducting sphere of radius R as shown in the figure.
a) Find the potential everywhere.
b) Find the induced surface charge on the sphere, as function of q. Integrate this to get the total induced charge.
c) Calculate the electrostatic energy of this configuration.

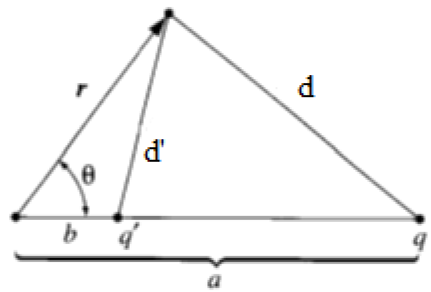


a) We can imagine a completely different configuration consisting of two charges, q and q'.
Where:

$$q' = -\frac{R}{a}q$$

Located at:

$$b = \frac{R^2}{a}$$



To the right of the center of the sphere as shown in the second figure.

Now the potential at a distance r from the origin due to these two point charges is:

$$V_p = \frac{1}{4\pi\varepsilon_o}\left[\frac{q}{d} + \frac{q'}{d'}\right] = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{d} - \frac{R}{ad'}\right]$$

Now this potential vanishes everywhere on the surface of the sphere (that was removed to place the image charge)

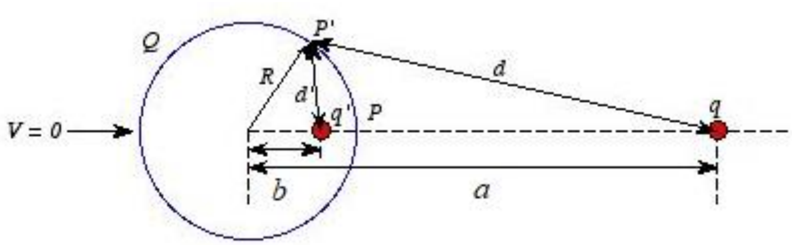
$$\vec{d} = \vec{r} - \vec{a} \text{ and } \vec{d}' = \vec{r} - \vec{b}$$

$$d = \sqrt{r^2 + a^2 - 2ar\cos\theta}$$

$$d' = \sqrt{r^2 + b^2 - 2br\cos\theta}$$

$$V_p = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{R}{a\sqrt{r^2 + b^2 - 2br\cos\theta}}\right]$$

When r=R on the surface of the sphere:



$$V_p = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{R^2 + a^2 - 2aR\cos\theta}} - \frac{R}{a\sqrt{R^2 + b^2 - 2bR\cos\theta}}\right]$$

Since

$$b = \frac{R^2}{a}$$

$$V_p = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{R^2 + a^2 - 2aR\cos\theta}} - \frac{R}{a\sqrt{R^2 + \frac{R^4}{a^2} - 2\frac{R^2}{a}R\cos\theta}}\right]$$

$$V_p = \frac{q}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{R^2 + a^2 - 2aR\cos\theta}} - \frac{R}{a\frac{R}{a}\sqrt{a^2 + R^2 - 2aR\cos\theta}}\right] = 0$$

Thus we conclude that the configuration of charge and image charge produces an electrostatic potential that is zero at any point on a sphere with radius R and centered at the origin.

Therefore, this charge configuration produces an electrostatic potential that satisfies exactly the same boundary conditions as the potential produced by the charge-sphere system.

The surface charge density σ on the sphere can be obtained from the boundary conditions of $\bar{E}$

$$\vec{E}_{outside} - \vec{E}_{inside} = \vec{E}_{outside} = \frac{\sigma}{\varepsilon_o}\hat{r}$$

$$\sigma = \varepsilon_o E_r = -\varepsilon_o\frac{\partial V}{\partial r}$$

Substituting the general expression for V into this equation we obtain

$$\sigma = -\varepsilon_o\frac{q}{4\pi\varepsilon_o}\frac{\partial}{\partial r}\left[\left[\frac{1}{\sqrt{r^2 + a^2 - 2ar\cos\theta}} - \frac{R}{a\sqrt{r^2 + b^2 - 2br\cos\theta}}\right]\right]_{r=R}$$

$$\sigma = -\frac{q}{4\pi}\left[-\frac{2r - 2a\cos\theta}{2(r^2 + a^2 - 2ar\cos\theta)^{\frac{3}{2}}} - \frac{-R * (2r - 2b\cos\theta)}{a * 2(r^2 + b^2 - 2br\cos\theta)^{\frac{3}{2}}}\right]_{r=R}$$

$$b = \frac{R^2}{a}$$

$$\sigma = -\frac{q}{4\pi}\left[-\frac{r - a\cos\theta}{(r^2 + a^2 - 2ar\cos\theta)^{\frac{3}{2}}} - \frac{-R * \left(r - \frac{R^2}{a}\cos\theta\right)}{a * \left(r^2 + \frac{R^4}{a^2} - \frac{2R^2}{a}r\cos\theta\right)^{\frac{3}{2}}}\right]_{r=R}$$

$$\sigma = -\frac{q}{4\pi}\left[-\frac{R - a\cos\theta}{(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}} - \frac{-R * \left(R - \frac{R^2}{a}\cos\theta\right)}{a * \left(R^2 + \frac{R^4}{a^2} - \frac{2R^3}{a}\cos\theta\right)^{\frac{3}{2}}}\right]$$

$$\sigma = -\frac{q}{4\pi}\left[-\frac{R - a\cos\theta}{(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}} - \frac{-R^2 * \left(1 - \frac{R}{a}\cos\theta\right)}{a * \frac{R^3}{a^3}(a^2 + R^2 - 2Ra\cos\theta)^{\frac{3}{2}}}\right]$$

$$\sigma = -\frac{q}{4\pi}\left[-\frac{R - a\cos\theta}{(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}} + \frac{a^2 - Ra\cos\theta}{R(a^2 + R^2 - 2Ra\cos\theta)^{\frac{3}{2}}}\right]$$

$$\sigma = -\frac{q}{4\pi}\left[\frac{-R^2 + aR\cos\theta - aR\cos\theta + a^2}{R(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}}\right] = -\frac{q}{4\pi R}\left[\frac{a^2 - R^2}{(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}}\right]$$

The total charge on the sphere can be obtained by integrating σ over the surface of the sphere. The result is

$$Q = \int \sigma da = \int -\frac{q}{4\pi R}\left[\frac{a^2 - R^2}{(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}}\right] R^2 \sin\theta \, d\theta \, d\phi$$

$$Q = -\frac{qR(a^2 - R^2)}{2}\int_0^{\pi} \frac{1}{(R^2 + a^2 - 2aR\cos\theta)^{\frac{3}{2}}} \sin\theta \, d\theta$$

$$\cos\theta = y \, , -\sin\theta \, d\theta = dy$$

$$= +\frac{qR(a^2 - R^2)}{2}\left[\frac{(R^2 + a^2 - 2aRy)^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)(-2aR)}\right]_1^{-1}$$

$$= \frac{qR(a^2 - R^2)}{2} * \frac{1}{aR}\left[\frac{1}{a + R} - \frac{1}{a - R}\right]$$

$$Q = \frac{q(a^2 - R^2)}{2} * \frac{1}{a}\left[\frac{-2R}{(a^2 - R^2)}\right] = -\frac{qR}{a} = q'$$

The force on q due to the sphere would be same as the force between q and the image charge:

$$F = \frac{1}{4\pi\varepsilon_o}\frac{qq'}{(a - b)^2} = \frac{1}{4\pi\varepsilon_o}\frac{q\left(\frac{qR}{a}\right)}{(a - R^2/a)^2} = \frac{1}{4\pi\varepsilon_o}\frac{q^2 Ra}{(a^2 - R^2)^2}$$

The total energy of the system would be to bring in the charge from infinity to point a.

$$W = \int_{\infty}^{a} \frac{1}{4\pi\varepsilon_o} \frac{q^2 R * r}{(r^2 - R^2)^2} dr$$

Let $r^2 = y$ then $2r\, dr = dy$, $r \to \infty$, $y \to \infty$ and for $r \to a$, $y \to a^2$

$$W = \frac{q^2 R}{8\pi\varepsilon_o} \int_{\infty}^{a^2} \frac{dy}{(y - R^2)^2} = \frac{q^2 R}{8\pi\varepsilon_o} * \left. \frac{(y - R^2)^{-1}}{-1} \right|_{\infty}^{a^2}$$

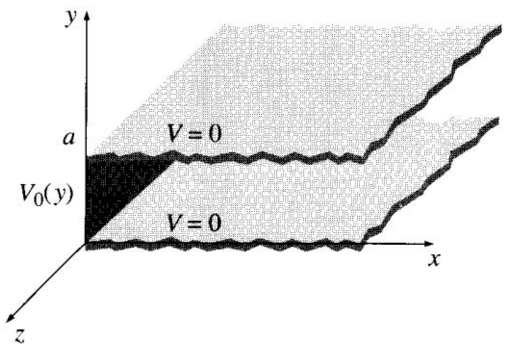
$$W = -\frac{1}{8\pi\varepsilon_o} \frac{Rq^2}{(a^2 - R^2)}$$

3.3. Separation of Variables

3.3.1. Separation of variables: Cartesian coordinates

A powerful technique very frequently used to solve partial differential equations is **separation of variables**, which is very helpful tool in solving partial differential equations. This method is particularly useful when the potential at the boundaries is known and we are to solve it within that region.

Example 5:
Two infinite, grounded, metal plates lie parallel to the *xz*-plane, one at $y = 0$, the other at $y = a$ as shown in the figure. The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates and maintained at a specified potential $V_o(y)$. Find the potential inside this "slot".



The electrostatic potential in the slot must satisfy the Laplace's equation. Since V is independent of z so we will use 2D Laplace's equation:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The boundary conditions are:

1. $V(x, y = 0) = 0$ (grounded bottom plate).

2. $V(x, y = a) = 0$ (grounded top plate).

3. $V(x = 0, y) = V_0(y)$ (plate at $x = 0$).

4. $V \to 0$ when $x \to \infty$.

These four boundary conditions specify the value of the potential on **all** boundaries surrounding the slot and are therefore sufficient to uniquely determine the solution of Laplace's equation inside the slot. Therefore, if we find one solution of Laplace's equation satisfying these boundary conditions than it must be the correct one.

Consider solutions of the following form: $V(x, y) = X(x)Y(y)$

If this is a solution of the two-dimensional Laplace equation than we must require that

$$\frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

$$Y(y)\frac{\partial^2 X(x)}{\partial x^2} + X(x)\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = 0$$

The first term of the left-hand side of this equation depends only on *x* while the second term depends only on *y*. Therefore, if this equation must hold for all *x* and *y* in the slot we must require that

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = C_1 = Constant$$

$$\frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} = C_2 = Constant$$

$$C_1 + C_2 = 0$$

The differential equation for *X* can be rewritten as:

$$\frac{d^2 X(x)}{dx^2} - C_1 X(x) = 0$$

If C_1 is a negative number then this equation can be rewritten as:

$$\frac{d^2 X(x)}{dx^2} + k^2 X(x) = 0$$

where $k^2 = -C_1$. The most general solution of this equation is

$$X(x) = A\cos(kx) + B\sin(kx)$$

However, this function is an oscillatory function and does not satisfy boundary condition # 4, which requires that *V* approaches zero when *x* approaches infinity. We therefore conclude that C_1 **cannot** be a negative number.

If C_1 is a positive number then the differential equation for *X* can be written as:

$$\frac{d^2 X(x)}{dx^2} - k^2 X(x) = 0$$

The most general solution of this equation is

$$X(x) = Ae^{kx} + Be^{-kx}$$

This solution will approach zero when *x* approaches infinity if $A = 0$. Thus

$$X(x) = Be^{-kx}$$

The solution for *Y* can be obtained by solving the following differential equation:

$$\frac{\partial^2 Y(y)}{\partial y^2} = -k^2 Y(y)$$

Since C_1 is positive C_2 has to be negative because $C_1 + C_2 = 0$

The most general solution of this equation is

$$Y(x) = C\sin(ky) + D\cos(ky)$$

Therefore, the general solution for the electrostatic potential *V*(*x*,*y*) is equal to

$$V(x,y) = X(x)Y(y) = e^{-kx}(C\sin(ky) + D\cos(ky))$$

where we have absorbed the constant *B* into the constants *C* and *D*. The constants *C* and *D* must be chosen such that the remaining three boundary conditions (1, 2, and 3) are satisfied.

The first boundary condition requires that $V(x, y = 0) = 0$:

$$V(x, y = 0) = e^{-kx}(C\sin(0) + D\cos(0)) = De^{-kx} = 0$$

which requires that $D = 0$. The second boundary condition requires that $V(x, y = a) = 0$:

$$V(x, y = a) = e^{-kx}(C\sin(ka)) = 0$$

which requires that $\sin(ka) = 0$. This condition limits the possible values of *k* to positive integers:

$$ka = n\pi \ ; \ k = \frac{n\pi}{a}, \text{ where } n = 1,2,3, \ldots$$

Note: negative values of *k* are not allowed as exp(-*kx*) approaches zero at infinity only if $k > 0$.

To satisfy boundary condition # 3 we must require that

$$V(x = 0, y) = C\sin(ky) = V_o(y)$$

This last expression suggests that the only time at which we can find a solution of Laplace's equation that satisfies all four boundary conditions has the form $e^{-kx}\sin(ky)$ when $V_o(y)$ happens to have the form $\sin(ky)$.

However, since *k* can take on an infinite number of values, there will be an infinite number of solutions to Laplace's equation satisfying boundary conditions # 1, # 2 and # 4. The most general form of the solution of Laplace's equation will be a linear superposition of all possible solutions. Thus

$$V(x,y)=\sum_{i=1}^{n} C_i e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)$$

Boundary condition # 3 can now be written as

$$V(x=0,y)=\sum_{i=1}^{n} C_i \sin\left(\frac{n\pi}{a}y\right)=V_o(y)$$

This is a Fourier sine series and we can use a mathematical trick to find the coefficients C_i.

Multiplying both sides by $\sin\left(\frac{n'\pi}{a}y\right)$ and integrating each side between $y = 0$ and $y = a$ we obtain

$$\sum_{i=1}^{n} C_i \int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right)dy=\int_0^a \sin\left(\frac{n'\pi}{a}y\right)\,V_o(y)dy$$

$$\int_0^a \sin\left(\frac{n'\pi}{a}y\right)\sin\left(\frac{n\pi}{a}y\right)dy=\begin{cases}0 & for \quad n'\neq n\\ \frac{a}{2} & for \quad n'=n\end{cases}$$

The integral on the left-hand side of this equation is equal to zero for all values of $n^{'}$ except for $n^{'}=n$. Thus

$$\sum_{i=1}^{n} C_i \int_0^a \sin\left(\frac{n'\pi}{a}y\right)\sin\left(\frac{n\pi}{a}y\right)dy=C_n\frac{a}{2}=\int_0^a \sin\left(\frac{n'\pi}{a}y\right)\,V_o(y)dy$$

$$C_n=\frac{2}{a}\int_0^a \sin\left(\frac{n'\pi}{a}y\right)\,V_o(y)dy$$

The coefficients C_n are called the **Fourier coefficients** of $V_o(y)$. The solution of Laplace's equation in the slot is therefore equal to

$$V(x,y)=\sum_{i=1}^{n} C_i e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)$$

where

$$C_n=\frac{2}{a}\int_0^a \sin\left(\frac{n'\pi}{a}y\right)\,V_o(y)dy$$

Now consider the special case where $V_o(y)=constant=V_o$, then the coefficient C_n would be:

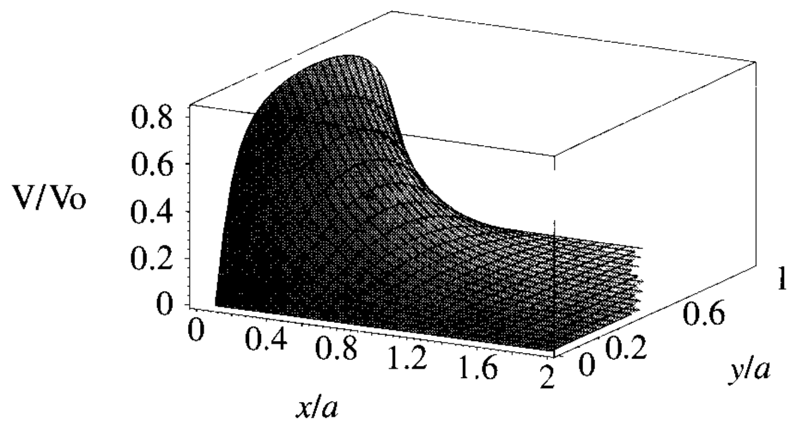
$$C_n=\frac{2}{a}V_o\int_0^a \sin\left(\frac{n'\pi}{a}y\right)\,dy$$

$$C_n=\frac{2}{n\pi}V_o(1-\cos(n\pi))=\begin{cases}0 \quad if \quad n=even\\ \frac{4V_o}{n\pi} \quad if \quad n=odd\end{cases}$$

And hence the solution to Laplace's equation will be:

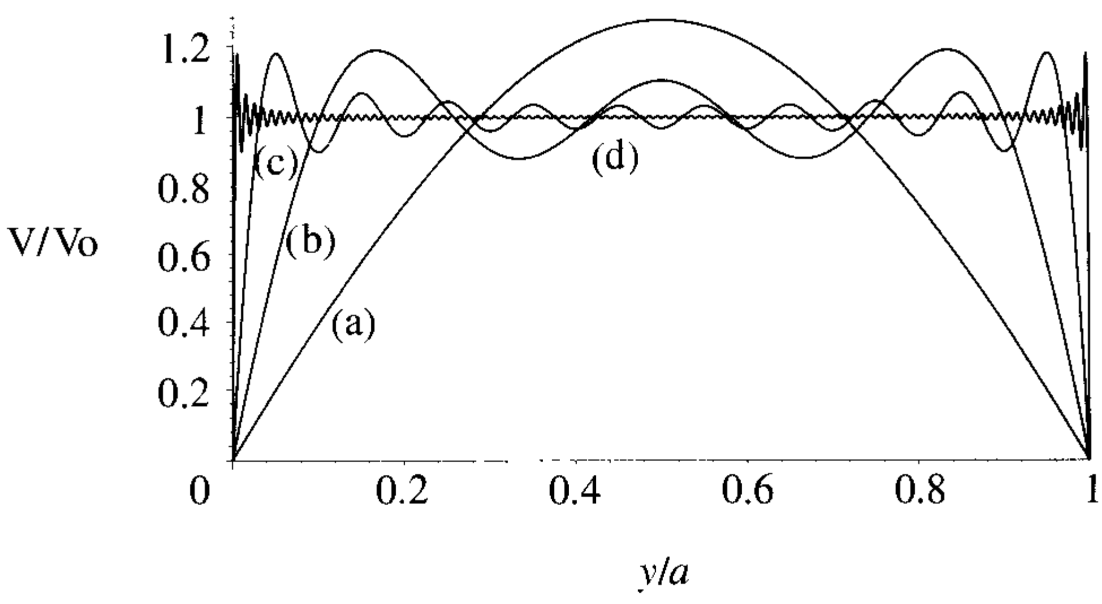
$$V(x,y) = \frac{4V_o}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{-\left(\frac{n\pi}{a}x\right)} \sin\left(\frac{n\pi}{a}y\right)$$

The figure below is the 2D plot of this solution:



The figure below shows how the first few terms in the Fourier series combine to make a better and better approximation to the constant V_o.

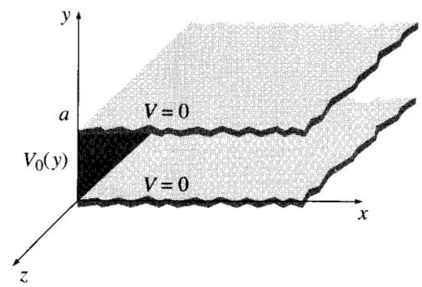
(a) is n=1 only, (b) include n up to 5, (c) includes n up to 10 and (d) include n up to 100.



Charge density on the strip at x=0

For the infinite slot determine the charge density $\sigma(y)$ on the strip at x=0, assuming it is a conductor at constant potential V_o.

The electrostatic potential in the slot is equal to



$$V(x,y) = \frac{4V_o}{\pi} \sum_{n=1,3,5,\dots} \frac{1}{n} e^{-\left(\frac{n\pi}{a}x\right)} \sin\left(\frac{n\pi}{a}y\right)$$

The charge density at the plate at x = 0 can be obtained using the boundary condition for the electric field at a boundary:

$$E_{x=0+} - E_{x=0-} = E_{x=0+} = \frac{\sigma}{\varepsilon_o}\hat{n}$$

where $\hat{n}$ is directed along the positive x axis. Since $\vec{E} = -\vec{\nabla}V$ this boundary condition can be rewritten as:

$$\left.\frac{\partial V}{\partial x}\right|_{x=0+} = -\frac{\sigma}{\varepsilon_o}$$

Differentiating V(x,y) with respect to x:

$$\frac{\partial V}{\partial x} = \frac{4V_o}{\pi} \sum_{i=1,3,5,\dots} \frac{1}{n}\left(-\frac{n\pi}{a}\right) e^{-\left(\frac{n\pi}{a}x\right)} \sin\left(\frac{n\pi}{a}y\right)$$

At the x = 0 boundary, we get:

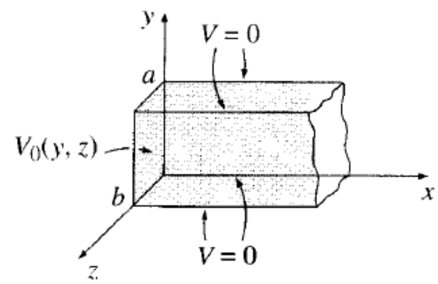
$$\left.\frac{\partial V}{\partial x}\right|_{x=0+} = -\frac{4V_o}{a} \sum_{n=1,3,5,\dots} \sin\left(\frac{n\pi}{a}y\right)$$

The charge density σ on the x = 0 strip is therefore equal to

$$\sigma = -\varepsilon_o \left.\frac{\partial V}{\partial x}\right|_{x=0+} = \frac{4V_o\varepsilon_o}{a} \sum_{n=1,3,5,\dots} \sin\left(\frac{n\pi}{a}y\right)$$

Example 6:

An infinite long rectangular metal pipe (sides a and b) is grounded but one end at x=0, is maintained at a specific potential $V_o(y,z)$ as shown in the figure below. Find the potential inside the pipe.



Solution:

Since there is no charge enclosed in the pipe, we can us Laplace's equation and solve for V:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

The boundary conditions are:

(i) V=0 when y=0
(ii) V=0 when y=a
(iii) V=0 when z=0
(iv) V=0 when z=b
(v) V=0 when $x = \infty$
(vi) $V = V_o(y,z)$ when $x = 0$

Let's assume the solution of the Laplace's equation is following:

$$V = X(x)Y(y)Z(z)$$

The Laplace's equation becomes

$$\frac{\partial^2}{\partial x^2}[X(x)Y(y)Z(z)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)Z(z)] + \frac{\partial^2}{\partial z^2}[X(x)Y(y)Z(z)] = 0$$

$$Y(y)Z(z)\frac{\partial^2 X(x)}{\partial x^2} + X(x)Z(z)\frac{\partial^2 Y(y)}{\partial y^2} + X(x)Y(y)\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = C_1$; $\frac{1}{Y(y)}\frac{\partial^2 Y(y)}{\partial y^2} = C_2$; $\frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = C_3$

With $C_1 + C_2 + C_3 = 0$

Similar to the previous example, boundary condition (v) suggests that C_1 has to be a positive constant, whereas C_2 and C_3 are negative.

$C_2 = -k^2$; $C_3 = -l^2$; and $C_1 = k^2 + l^2$

$$\frac{d^2X(x)}{dx^2} = (k^2 + l^2)X(x) \; ; \frac{d^2Y(y)}{dy^2} = k^2Y(y) \; ; \frac{d^2Z(z)}{dz^2} = l^2Z(z)$$

$$X(x) = Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x}$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$

$$Z(z) = E\sin(lz) + F\cos(lz)$$

a) Boundary condition (v) [V=0 when $x = \infty$] gives that A=0.
b) Boundary condition (i) [V=0 when $y = 0$] gives that D=0.
c) Boundary condition (iii) [V=0 when $z = 0$] gives that F=0.
d) Boundary condition (ii) [V=0 when y=a] gives that $k = \frac{n\pi}{a}$.
e) Boundary condition (iv) [V=0 when z=a] gives that $l = \frac{m\pi}{b}$.

So our solution reduces to:

$$X(x) = Be^{-\sqrt{k^2+l^2}x}$$

$$Y(y) = C\sin(ky)$$

$$Z(z) = E\sin(lz)$$

Or

$$V(x,y,z) = Ce^{-\pi\left(\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\right)x}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

The general solution will be the linear combination of all the possible values of n and m:

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}e^{-\pi\left(\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\right)x}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

The last boundary condition (vi) [$V = V_o(y,z)$ when $x = 0$] implies that:

$$V(0,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right) = V_o(y,z)$$

To determine C_{nm}, let's multiply the above expression with $\sin\left(\frac{n'\pi}{a}y\right)$ and $\sin\left(\frac{m'\pi}{b}z\right)$ and integrate:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}\int_0^a\int_0^b \sin\left(\frac{n'\pi}{a}y\right)\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m'\pi}{b}z\right)\sin\left(\frac{m\pi}{b}z\right)\,dy\,dz$$
$$=\int_0^a\int_0^b V_o(y,z)\sin\left(\frac{n'\pi}{a}y\right)\sin\left(\frac{m'\pi}{b}z\right)\,dy\,dz$$

$$\int_0^a \sin\left(\frac{n'\pi}{a}y\right)\sin\left(\frac{n\pi}{a}y\right)dy=\begin{cases}0 & for \quad n'\neq n\\ \frac{a}{2} & for \quad n'=n\end{cases}$$

$$\int_0^b \sin\left(\frac{m'\pi}{b}y\right)\sin\left(\frac{m\pi}{b}y\right)dy=\begin{cases}0 & for \quad m'\neq m\\ \frac{b}{2} & for \quad m'=m\end{cases}$$

$$C_{nm}*\frac{a}{2}*\frac{b}{2}=\int_0^a\int_0^b V_o(y,z)\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)\,dy\,dz$$

$$C_{nm}=\frac{4}{ab}\int_0^a\int_0^b V_o(y,z)\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)\,dy\,dz$$

So

$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{nm}e^{-\pi\left(\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\right)x}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$ along with the constant C_{nm} is the solution of our problem.

In case $V_o(y,z)=constant=V_o$, then we can find C_{nm} as:

$$C_{nm}=\frac{4V_o}{ab}\int_0^a \sin\left(\frac{n\pi}{a}z\right)dy\int_0^b \sin\left(\frac{m\pi}{b}z\right)dz=\begin{cases}0 & if\ n\ or\ m\ are\ even\\ \frac{16V_0}{\pi^2 nm} & if\ n\ and\ m\ are\ odd\end{cases}$$

So

$$V(x,y,z)=\sum_{n,m=1,3,5,\ldots}\frac{16V_0}{\pi^2 nm}e^{-\pi\left(\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\right)x}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

Notice that successive terms decrease rapidly due to $e^{-\pi\left(\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\right)x}$ term, so reasonable approximation would be to keep only the first few terms.

3.3.2. Separation of variables: spherical coordinates

For a spherical symmetric system, we can solve Laplace's equation using spherical coordinates.

Assuming the system has azimuthal symmetry $\left(\frac{\partial V}{\partial \phi}=0\right)$ Laplace's equation would be:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

Multiplying both sides by r^2 we get:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

Let's consider the solution of Laplace's equation is a function of r and θ, such that:

$$V(r,\theta)=R(r)\Theta(\theta)$$

Substituting this "solution" into Laplace's equation we obtain

$$\Theta(\theta)\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

Dividing each term of this equation by $R(r)\Theta(\theta)$ we get:

$$\frac{1}{\mathrm{R(r)}}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta(\theta)\ \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

The first term in this expression depends only on the distance r while the second term depends only on the angle θ. This equation can only be true for all r and θ if:

$$\frac{1}{\mathrm{R(r)}}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=l(l+1)$$

$$\frac{1}{\Theta(\theta)\ \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=-l(l+1)$$

and

Consider a solution for R of the following form:

$$\frac{d}{dr}\left(r^2\frac{\partial R}{\partial r}\right)=l(l+1)R(r)$$

This equation has a general solution:

$$R(r)=Ar^l+\frac{B}{r^{l+1}}$$

Similarly angular equation can be written as:

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\,\Theta(\theta)$$

The solution to this equation are the Legendre polynomials in the variable $\cos\theta$

$$\Theta(\theta) = P_l(\cos\theta)$$

Where $P_l(x)$ are defined by Rodrigues formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

The first few Legendre polynomials are written as:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = (3x^2-1)/2$$

$$P_3(x) = (5x^3-3x)/2$$

$$P_4(x) = (35x^4-30x^2+3)/8$$

$$P_5(x) = (63x^5-70x^3+15x)/8$$

So the most general solution for Laplace's equation can be written as:

$$V(r,\theta) = R(r)\Theta(\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l\cos(\theta)$$

Example 7:
The potential at the surface of a sphere is given by $V_o(\theta) = k\cos(3\theta)$, where k is some constant. Find the potential inside and outside the sphere, as well as the surface charge density $\sigma(\theta)$ on the sphere. (Assume that there is no charge inside or outside of the sphere.)

The most general solution of Laplace's equation in spherical coordinates is

$$V(r,\theta) = R(r)\Theta(\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l\cos(\theta)$$

First consider the region inside the sphere ($r < R$). In this region $B_l = 0$ otherwise potential would be infinity at $r = 0$. Thus

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l\, P_l\cos(\theta)$$

The potential at $r = R$ is therefore equal to

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l R^l\, P_l\cos(\theta) = k\cos(3\theta)$$

Using trigonometric relations, we can rewrite $\cos(3\theta)$ as

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta = \frac{8}{5}P_3(\cos\theta) - \frac{3}{5}P_1(\cos\theta)$$

Substituting this in the above equation for $V(r,\theta)$:

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l R^l\, P_l\cos(\theta) = k\cos(3\theta) = \frac{8k}{5}P_3(\cos\theta) - \frac{3k}{5}P_1(\cos\theta)$$

This equation immediately shows that $A_l = 0$ except for $l = 1\ or\ 3$.

So $A_1 = -\frac{3k}{5R}$ and $A_3 = \frac{8k}{5R^3}$

The electrostatic potential inside the sphere is therefore equal to

$$V(r,\theta) = \frac{8k}{5R^3}r^3 P_3(\cos\theta) - \frac{3k}{5R}rP_1(\cos\theta)$$

Now consider the region outsider the sphere ($r > R$). In this region $A_l = 0$ otherwise $V(r,\theta)$ would be infinity at $r = \infty$.

Hence

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l\cos(\theta)$$

The potential at $r = R$ is therefore equal to

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l \cos(\theta) = \frac{8k}{5} P_3(\cos\theta) - \frac{3k}{5} P_1(\cos\theta)$$

This implies that $B_l = 0$ except when $l = 1\ or\ 3$, which gives:

$$B_1 = -\frac{3k}{5} R^2$$
$$B_3 = \frac{8k}{5} R^4$$

The electrostatic potential outside the sphere is thus equal to

$$V(r,\theta) = \frac{8k}{5r^4} R^4 P_3(\cos\theta) - \frac{3k}{5r^2} R^2 P_1(\cos\theta)$$

The charge density on the sphere can be obtained using the boundary conditions for the electric field at a boundary:

$$\vec{E}_{r=R+} - \vec{E}_{r=R+} = \frac{\sigma(\theta)}{\varepsilon_o} \hat{r}$$

Since $\vec{E} = -\vec{\nabla} V$ this boundary condition can be rewritten as:

$$\left.\frac{\partial V}{\partial r}\right|_{r=R+} - \left.\frac{\partial V}{\partial r}\right|_{r=R+} = -\frac{\sigma(\theta)}{\varepsilon_o}$$

The first term on the left-hand side of this equation can be calculated using the electrostatic potential just obtained:

$$\left.\frac{\partial V}{\partial r}\right|_{r=R+} = \left[-\frac{32k}{5r^5} R^4 P_3(\cos\theta) + \frac{6k}{5r^3} R^2 P_1(\cos\theta)\right]_{r=R+} = \frac{k}{5R}\left(6P_1(\cos\theta) - 32P_3(\cos\theta)\right)$$

In the same manner we obtain

$$\left.\frac{\partial V}{\partial r}\right|_{r=R-} = \frac{\partial}{\partial r}\left(\frac{8k}{5R^3} r^3 P_3(\cos\theta) - \frac{3k}{5R} r P_1(\cos\theta)\right) = \left[\frac{24k}{5R^3} r^2 P_3(\cos\theta) - \frac{3k}{5R} P_1(\cos\theta)\right]_{r=R-}$$
$$\left.\frac{\partial V}{\partial r}\right|_{r=R-} = \left(\frac{24k}{5R} P_3(\cos\theta) - \frac{3k}{5R} P_1(\cos\theta)\right) = \frac{k}{5R}[24P_3(\cos\theta) - 3P_1(\cos\theta)]$$

So,

$$\left.\frac{\partial V}{\partial r}\right|_{r=R+} - \left.\frac{\partial V}{\partial r}\right|_{r=R-} = \frac{k}{5R}\left(6P_1(\cos\theta) - 32P_3(\cos\theta)\right) - \frac{k}{5R}[24P_3(\cos\theta) - 3P_1(\cos\theta)]$$

$$\sigma(\theta) = -\frac{k\varepsilon_o}{5R}[9P_1(\cos\theta) - 56P_3(\cos\theta)]$$

Laplace's equation in Cylindrical Coordinates:
Solve Laplace's equation by separation of variables in *cylindrical* coordinates, assuming there is no dependence on z (cylindrical symmetry). Make sure that you find *all* solutions to the radial equation. Does your result accommodate the case of an infinite line charge?

For a system with cylindrical symmetry the electrostatic potential does not depend on z. This immediately implies that $\frac{\partial V}{\partial z} = 0$. Under this assumption Laplace's equation reads

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Consider as a possible solution of V:

$$V(r,\phi) = R(r)\alpha(\phi)$$

Substituting this solution into Laplace's equation we get:

$$\frac{\alpha(\phi)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{R(r)}{r^2}\frac{\partial^2 \alpha(\phi)}{\partial \phi^2} = 0$$

Multiplying each term in this equation by r^2 and dividing by $R(r)\alpha(\phi)$ we get:

$$\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{\alpha(\phi)}\frac{\partial^2 \alpha(\phi)}{\partial \phi^2} = 0$$

$$\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = constant = \gamma \qquad \text{and} \qquad \frac{1}{\alpha(\phi)}\frac{\partial^2 \alpha(\phi)}{\partial \phi^2} = -\gamma$$

First consider the case in which $= -m^2 < 0$. The differential equation for $\alpha(\phi)$ can be rewritten as

$$\frac{\partial^2 \alpha(\phi)}{\partial \phi^2} - m^2\alpha(\phi) = 0$$

The most general solution of this differential solution is:

$$\alpha_m(\phi) = C_m e^{m\phi} + D_m e^{-m\phi}$$

However, in cylindrical coordinates we require that any solution for a given ϕ is equal to the solution for $\phi + 2\pi$. Obviously this condition is not satisfied for this solution, and we conclude that $= m^2 \geq 0$. The differential equation for $\alpha(\phi)$ can be rewritten as:

$$\frac{\partial^2 \alpha(\phi)}{\partial \phi^2} + m^2\alpha(\phi) = 0$$

The most general solution of this differential solution is:

$$\alpha_m(\phi) = C_m \cos(m\phi) + D_m \sin(m\phi)$$

The condition that $\alpha(\phi) = \alpha(\phi + 2\pi)$ requires that m is an integer.

Now consider the radial function $R(r)$:

$$\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = constant = \gamma = m^2 > 0$$

The general solution for this equation would be:

$$R(r) = Ar^k$$

Substituting this solution into the previous differential equation we get:

$$\frac{r}{Ar^k}\frac{\partial}{\partial r}\left(r\frac{\partial(Ar^k)}{\partial r}\right) = \frac{1}{Ar^{k-1}}\frac{\partial}{\partial r}(Akr^k) = \frac{1}{Ar^{k-1}}k^2Ar^{k-1} = k^2 = m^2$$

$$\frac{r}{Ar^k}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(Ar^k\right)\right) = \frac{1}{Ar^{k-1}}\frac{\partial}{\partial r}\left(r\left(kAr^{k-1}\right)\right) = \frac{1}{Ar^{k-1}}\frac{\partial}{\partial r}\left(kAr^k\right) = \frac{1}{Ar^{k-1}}k^2Ar^{k-1} = k^2 = m^2$$

Therefore, the constant k can take on the following two values:

$$k_+ = m \text{ and } k_- = -m$$

The most general solution for $R(r)$ under the assumption that $m^2 > 0$ is therefore

$$R_m(r) = A_m r^m + \frac{B_m}{r^m}$$

Now consider the solutions for $R(r)$ when $m^2 = 0$. In this case we require that:

$$\frac{r}{R(r)}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = constant = \gamma = m^2 = 0 \quad \text{or} \quad \frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = 0$$

Which requires: $r\frac{\partial R}{\partial r} = constan = a_o$ which implies that:

$$\frac{\partial R}{\partial r} = \frac{a_o}{r}$$

If $a_o = 0$ then the solution of this differential equation is

$$R(r) = b_o = constant$$

If $a_o \neq 0$ then the solution of this differential equation is

$$R(r) = a_o \ln(r) + b_o$$

Combining the solutions obtained for $m^2 = 0$ with the solutions obtained for $m^2 > 0$, we conclude that the most general solution for $R(r)$ is:

$$R(r) = a_o \ln(r) + b_o + \sum_{m=1}^{\infty}\left[A_m r^m + \frac{B_m}{r^m}\right]$$

Therefore, the most general solution of Laplace's equation for a system with cylindrical symmetry is

$$V(r,\phi) = a_o \ln(r) + b_o + \sum_{m=1}^{\infty}\left[\left(A_m r^m + \frac{B_m}{r^m}\right)\left(C_m \cos(m\phi) + D_m \sin(m\phi)\right)\right]$$

Example 8:

A charge density $\sigma = a\, sin(5\phi)$ is glued over the surface of an infinite cylinder of radius R. Find the potential inside and outside the cylinder.

The electrostatic potential can be obtained using the general solution of Laplace's equation for a system with cylindrical symmetry. In the region inside the cylinder the coefficient B_m =o otherwise $V(r,\phi) \to \infty$ at $r = 0$, and for the same reason $a_o = 0$.

So the general solution to Laplace equation will reduce to:

$$V_{in}(r,\phi) = b_{o,in} + \sum_{m=1}^{\infty}\left[r^m\left(C_{m,in}\cos(m\phi) + D_{m,in}\sin(m\phi)\right)\right]$$

In the region outside the cylinder the coefficients $A_m = 0$ and $a_o = 0$. Thus

$$V_{out}(r,\phi) = b_{o,out} + \sum_{m=1}^{\infty}\left[\frac{1}{r^m}\left(C_{m,out}\cos(m\phi) + D_{m,out}\sin(m\phi)\right)\right]$$

Since $V(r,\phi)$ must approach 0 when *r* approaches infinity, we must also require that $b_{o,out} = 0$.

$$V_{out}(r,\phi) = \sum_{m=1}^{\infty}\left[\frac{1}{r^m}\left(C_{m,out}\cos(m\phi) + D_{m,out}\sin(m\phi)\right)\right]$$

The charge density on the surface of the cylinder is equal to

$$\sigma(\phi) = -\varepsilon_o\left[\left.\frac{\partial V}{\partial r}\right|_{r=R_+} - \left.\frac{\partial V}{\partial r}\right|_{r=R_-}\right]$$

Differentiating $V(r,\phi)$ in the region *r* > *R* and setting *r* = *R* we obtain

$$\left.\frac{\partial V}{\partial r}\right|_{r=R_+} = \sum_{m=1}^{\infty}\left[-\frac{m}{R^{m+1}}\left(C_{m,out}\cos(m\phi) + D_{m,out}\sin(m\phi)\right)\right]$$

Differentiating $V(r,\phi)$ in the region *r* < *R* and setting *r* = *R* we obtain

$$\left.\frac{\partial V}{\partial r}\right|_{r=R_+} = \sum_{m=1}^{\infty}\left[-mR^{m-1}\left(C_{m,in}\cos(m\phi) + D_{m,in}\sin(m\phi)\right)\right]$$

The charge density on the surface of the cylinder is therefore equal to

$$\sigma(\phi) = \varepsilon_o\sum_{m=1}^{\infty}\left[\left(\frac{m}{R^{m+1}}C_{m,out} + mR^{m-1}C_{in,m}\right)\cos(m\phi) + \left(\frac{m}{R^{m+1}}D_{m,out} + mR^{m-1}D_{in,m}\right)\sin(m\phi)\right]$$

Since the charge density is proportional to $\sin(5\phi)$ we can conclude immediately that $C_{in,m} = C_{out,m} = 0$ for all m and that $D_{in,m} = D_{out,m} = 0$ for all m except m=5.

Therefore the charge density is:

$$\sigma(\phi) = \varepsilon_o \left(\frac{5}{R^6} D_{5,out} + 5R^4 D_{in,5} \right) \sin(5\phi) = a \sin(5\phi)$$

$$\frac{5}{R^6} D_{5,out} + 5R^4 D_{in,5} = \frac{a}{\varepsilon_o}$$

A second relation between $D_{in,5}$ and $D_{out,5}$ can be obtained using the condition that the electrostatic potential is continuous at any boundary. This requires that

$$V_{in}(r,\phi) = V_{out}(r,\phi)$$

$$b_{o,in} + R^5 D_{5,in} \sin(5\phi) = \frac{1}{R^5} D_{5,out} \sin(5\phi)$$

Thus, $b_{in,0} = 0$ and $D_{out,5} = R^{10} D_{in,5}$

We now have two equations with two unknown, $D_{in,5}$ and $D_{out,5}$, which can be solved with the following result:

$$D_{in,5} = \frac{a}{10\varepsilon_o} \frac{1}{R^4} \qquad \text{and} \qquad D_{out,5} = \frac{a}{10\varepsilon_o} R^6$$

The electrostatic potential inside the cylinder is thus equal to

$$V_{in}(r,\phi) = r^5 D_{5,in} \sin(5\phi)$$

$$V_{in}(r,\phi) = \frac{a}{10\varepsilon_o} \frac{r^5}{R^4} \sin(5\phi)$$

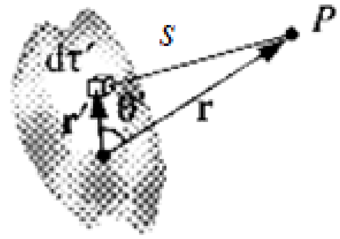
The electrostatic potential outside the cylinder is thus equal to

$$V_{out}(r,\phi) = \frac{D_{5,out}}{r^5} \sin(5\phi)$$

$$V_{out}(r,\phi) = \frac{a}{10\varepsilon_o} \frac{R^6}{r^5} \sin(5\phi)$$

3.4. Multipole Expansions

Consider a given charge distribution ρ as shown in the figure and the potential at point P is:



$$V(P) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(r')}{s}\, d\tau'$$

where s is the distance between P and an infinitesimal segment of the charge distribution. We can write d:

$$s^2 = r^2 + r'^2 - 2rr'\cos\theta = r^2\left(1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right)$$

$s = r\sqrt{(1+\varepsilon)}$ where $\varepsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$

This equation can be rewritten as

$$\frac{1}{s} = \frac{1}{r}\frac{1}{\sqrt{1+\varepsilon}}$$

At large distances from the charge distribution $r \gg r'$ and consequently $\frac{r'}{r} \ll 1$. Using the binomial expansion:

$\left[(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k = \sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x^k = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\right]$:

$$\frac{1}{\sqrt{1+\varepsilon}} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots$$

we can rewrite $1/d$ as

$$\frac{1}{s} \approx \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left[\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)\right]^2 - \frac{5}{16}\left[\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)\right]^3 + \cdots\right]$$

$$\frac{1}{s} \approx \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \cdots\right] = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

Where θ' is the angle between r and r',

Using this expansion of $1/d$ we can rewrite the electrostatic potential at P as

$$V(P) = \frac{1}{4\pi\varepsilon_o}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int \rho(r')r'^n P_n(\cos\theta')\, d\tau'$$

This expression is valid for all r (not only $\gg r'$). However, if $r \gg r'$ then the potential at P will be dominated by the first non-zero term in this expansion. This expansion is known as the **multipole expansion**. In the limit of $r \gg r'$ only the first terms in the expansion need to be considered:

$$V(P) = \frac{1}{4\pi\varepsilon_o}\left[\frac{1}{r}\int \rho(r')\, d\tau' + \frac{1}{r^2}\int \rho(r')\, r'\cos\theta'\, d\tau' + \frac{1}{r^3}\int \rho(r')r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\, d\tau' + \cdots\right]$$

The first term in this expression, proportional to $1/r$, is called the **monopole term**. The second term in this expression, proportional to $1/r^2$, is called the **dipole term**. The third term in this expression, proportional to $1/r^3$, is called the **quadrupole term**.

3.4.1. The monopole term.

If the total charge of the system is non zero then the electrostatic potential at large distances is dominated by the monopole term:

$$V(P) = \frac{1}{4\pi\varepsilon_o}\frac{1}{r}\int \rho(r')\,d\tau' = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r}$$

where Q is the total charge of the charge distribution.

The electric field associated with the monopole term can be obtained by calculating the gradient of $V(P)$:

$$\vec{E}(P) = -\vec{\nabla}V(P) = -\frac{1}{4\pi\varepsilon_o}Q\vec{\nabla}\left(\frac{1}{r}\right) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$$

3.4.2. The dipole term.

If the total charge of the charge distribution is equal to zero ($Q = 0$) then the monopole term in the multipole expansion will be equal to zero. In this case, the dipole term will dominate the electrostatic potential at large distances

$$V(P) = \frac{1}{4\pi\varepsilon_o}\frac{1}{r^2}\int \rho(r')\,r'\cos\theta'\,d\tau'$$

Since θ' is the angle between $\bar{r}$ and $\bar{r}'$ we can rewrite $r'\cos\theta$ as

$$r'\cos\theta' = \hat{r}.\vec{r}'$$

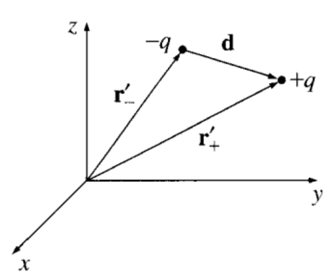
The electrostatic potential at P can therefore be rewritten as

$$V(P) = \frac{1}{4\pi\varepsilon_o}\frac{\hat{r}}{r^2}.\int \rho(r')\,\vec{r}'d\tau' = \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}.\hat{r}}{r^2}$$

In this expression $\bar{P}$ is the **dipole moment** of the charge distribution which is defined as

$$\vec{p} = \int \rho(r')\,\vec{r}'d\tau'$$

The dipole moment depends on the geometry (size, shape, and density) of the charge distribution. Similar expression for the dipole moment can be written for point, line and surface charge distributions as well.



For collection of point charges:

$$\vec{p} = \sum_{i=1}^{n} q_i r_i'$$

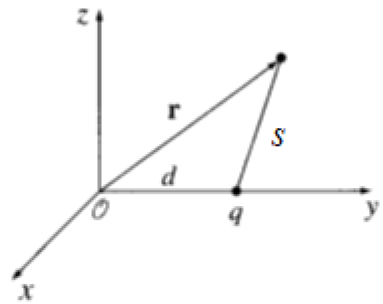
For the physical dipole consisting of a $\pm q$,

$$\vec{p} = q\vec{r}_+' - q\vec{r}_-' = q(\vec{r}_+' - \vec{r}_-') = q\vec{d}$$

Where $\vec{d}$ is a vector from the negative charge to the positive charge.

Origin of Coordinates in Multipole Expansion:

A point charge at the origin constitutes a pure monopole but if the point charge is not at the origin of a coordinates system then it is no longer a pure monopole.



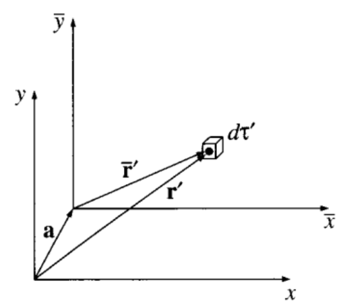
For example, charge in the figure below has a dipole moment $\vec{p} = qd\hat{\jmath}$ so there will be a dipole term in its potential.

The monopole potential $\left(\frac{1}{4\pi\varepsilon_o}\frac{q}{r}\right)$ is not correct for this configuration rather the potential would be: $\left(\frac{1}{4\pi\varepsilon_o}\frac{q}{s}\right)$

When we expand s in terms of r we will get all kind of powers not just the first power.

The monopole term will not change because the total charge is independent of the coordinate system but dipole and higher moments will change.

Let's say if the origin is shifted by amount a as shown in the figure then:



$$\vec{\bar{p}} = \int \vec{\bar{r}}'\rho(\vec{r}')d\tau' = \int \vec{\bar{r}}'\rho(\vec{r}')d\tau' = \int (\vec{r}' - \vec{a})\rho(\vec{r}')d\tau'$$

$$\vec{\bar{p}} = \int \vec{r}'\rho(\vec{r}')d\tau' - \vec{a}\int \rho(\vec{r}')d\tau' = \vec{p} - \vec{a}Q$$

$$\vec{\bar{p}} = \vec{p} - \vec{a}Q$$

If the total charge Q=0 then $\vec{\bar{p}} = \vec{p}$, as in figure(a) below where total charge is zero so dipole moment is simply $q\vec{d}$.

But in the case of figure (b), the total charge is not zero, so dipole moment will depend on the origin we choose.

(a) (b)

<u>The electric field of a Dipole</u>

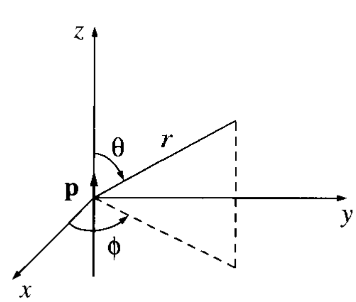
If we choose a coordinate system where $\vec{p}$ lies at the origin and points in the z-direction, then the electric field associated with the dipole term can be obtained by calculating the gradient of $V(P)$:

$$V(P) = \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}.\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_o}\frac{p\cos\theta}{r^2}$$

$$E_r(P) = -\frac{\partial V(P)}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_o}\frac{1}{r^3}; \quad E_\theta(P) = -\frac{1}{r}\frac{\partial V(P)}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_o}\frac{1}{r^3};$$

$$E_\phi(P) = -\frac{1}{r\sin\theta}\frac{\partial V(P)}{\partial\phi} = 0$$

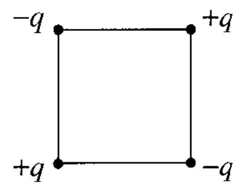
$$\vec{E}_{dipole}(r,\theta) = \frac{1}{4\pi\varepsilon_o}\frac{p}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\ \hat{\theta})$$



Dipole moments are vectors and they add vectorially, for example if there are two dipoles with dipole moments $\vec{p}_1$ and $\vec{p}_2$ then the net dipole moment of the system would be:

$$\vec{p}_{net} = \vec{p}_1 + \vec{p}_2$$

In the figure there are four charges shown on the corner of a square, what is the net dipole moment for this arrangement:



$$\vec{p}_{net} = 0$$

$$\uparrow + \downarrow = 0 \qquad \text{or} \qquad \rightarrow + \leftarrow = 0$$

Example 9: A "pure" dipole p is situated at the origin, pointing in the z-direction.

(a) What is the force on a point charge q at *(a,0,0)* (Cartesian coordinates)?
(b) What is the force on q at *(0,0,a)*?
(c) How much work does it take to move charge q from *(a,0,0)* to *(0,0,a)*?

Solution:

(a) The charge q is locate at $r = a$ and $\theta = \pi/2$, so

$$\vec{E}_{dipole}(r,\theta) = \frac{1}{4\pi\varepsilon_o}\frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\ \hat{\theta}\right) = \frac{1}{4\pi\varepsilon_o}\frac{p}{a^3}\hat{\theta}$$

$$\left.\begin{aligned} \hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}, \end{aligned}\right\}$$

$$\vec{E}_{dipole}(x,y,z) = \frac{1}{4\pi\varepsilon_o}\frac{p}{a^3}(-\hat{z}) = -\frac{1}{4\pi\varepsilon_o}\frac{p}{a^3}\hat{z}$$

$$\vec{F} = q\vec{E} = -\frac{1}{4\pi\varepsilon_o}\frac{qp}{a^3}\hat{z}$$

(b) The charge q is locate at $r = a$ and $\theta = 0$, so

$$\vec{E}_{dipole}(r,\theta) = \frac{1}{4\pi\varepsilon_o}\frac{p}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\ \hat{\theta}\right) = \frac{1}{2\pi\varepsilon_o}\frac{p}{a^3}\hat{r} = \frac{1}{2\pi\varepsilon_o}\frac{p}{a^3}\hat{z}$$

$$\vec{F} = q\vec{E} = \frac{1}{2\pi\varepsilon_o}\frac{qp}{a^3}\hat{z}$$

(c) $W = q\Delta V = q[V(0,0,a) - V(a,0,0)] = q\left[\frac{1}{4\pi\varepsilon_o}\frac{p\cos 0}{a^2} - \frac{1}{4\pi\varepsilon_o}\frac{p\cos(\pi/2)}{a^2}\right] = \frac{1}{4\pi\varepsilon_o}\frac{qp}{a^2}$

Example 10:

A thin insulating rod, running from z=-a to z=+a, carries the following line charges:

(a) $\lambda = \lambda_o \cos\left(\frac{\pi z}{2a}\right)$
(b) $\lambda = \lambda_o \sin\left(\frac{\pi z}{a}\right)$
(c) $\lambda = \lambda_o \cos\left(\frac{\pi z}{a}\right)$

In each case find the leading term in the multipole expansion.

Solution:
a) The total charge on the rod is equal to

$$Q_{tot} = \int_{-a}^{+a} \lambda dz = \int_{-a}^{+a} \lambda_0 \cos\left(\frac{\pi z}{2a}\right) dz = \frac{4a}{\pi} \lambda_0$$

Since $Q_{tot} \neq 0$, the monopole term will dominate the electrostatic potential at large distances. Thus

$$V_P = \frac{1}{4\pi\varepsilon_0} \frac{4a}{\pi} \lambda_0 \frac{1}{r}$$

b) The total charge on the rod is equal to zero. Therefore, the electrostatic potential at large distances will be dominated by the dipole term (if non-zero). The dipole moment of the rod is equal to

$$p = \int_{-a}^{+a} z\lambda dz = \int_{-a}^{+a} z\lambda_0 \sin\left(\frac{\pi z}{a}\right) dz = \frac{2a^2}{\pi} \lambda_0$$

Since the dipole moment of the rod is not equal to zero, the dipole term will dominate the electrostatic potential at large distances. Therefore

$$V_P = \frac{1}{4\pi\varepsilon_0} \frac{2a^2}{\pi} \lambda_0 \frac{1}{r^2} \cos\theta$$

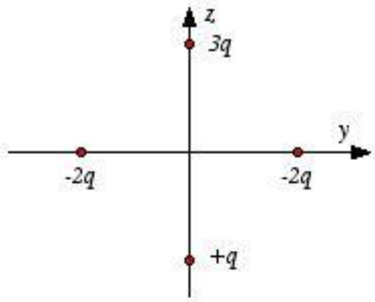
c) For this charge distribution the total charge is equal to zero and the dipole moment is equal to zero. The electrostatic potential of this charge distribution is dominated by the quadrupole term.

$$I_2 = \int_{-a}^{+a} z^2 \lambda dz = \int_{-a}^{+a} z^2 \lambda_0 \cos\left(\frac{\pi z}{a}\right) dz = -\frac{4a^3}{\pi^2} \lambda_0$$

The electrostatic potential at large distance from the rod will be equal to

$$V_P = \frac{1}{4\pi\varepsilon_0} \left(-\frac{4a^3}{\pi^2} \lambda_0\right) \frac{1}{r^3} \frac{1}{2} \left(3\cos^2\theta - 1\right)$$

Example 11:
Four particles (one of charge *q*, one of charge 3*q*, and two of charge -2*q*) are placed as shown in Figure 3.12, each a distance *d* from the origin. Find a simple approximate formula for the electrostatic potential, valid at a point *P* far from the origin.



Solution:
The total charge of the system is equal to zero and therefore the monopole term in the multipole expansion is equal to zero. The dipole moment of this charge distribution is equal to

$$\bar{p} = \sum_i q_i \bar{r}_i = (-2q)d\,\hat{j} + (q)(-d)\,\hat{k} + (-2q)(-d)\,\hat{j} + (3q)d\,\hat{k} = 2qd\,\hat{k}$$

The Cartesian coordinates of *P* are

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

The scalar product between $\bar{p}$ and $\hat{r}$ is therefore

$$\bar{p} \bullet \hat{r} = 2qd\cos\theta$$

The electrostatic potential at *P* is therefore equal to

$$V_P = \frac{1}{4\pi\varepsilon_0}\frac{\bar{p}\bullet\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{2qd\cos\theta}{r^2}$$

Example 12:

A charge Q is distributed uniformly along the z axis from $z = -a$ to $z = a$. Show that the electric potential at a point (r, θ) is given by

$$V(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 + \frac{1}{3}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{5}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) + \ldots\right)$$

for $r > a$.

The charge density along this segment of the z axis is equal to

$$\rho = \frac{Q}{2a}$$

Therefore, the nth moment of the charge distribution is equal to

$$I_n = \int_{-a}^{a} z^n \rho dz = \frac{Q}{2a}\int_{-a}^{a} z^n dz = \frac{Q}{2a}\frac{z^{n+1}}{n+1}\bigg[_{-a}^{a} = \frac{Q}{2a}\frac{a^{n+1}}{n+1}\left\{1-(-1)^{n+1}\right\} = \frac{Q}{2}\frac{a^n}{n+1}\left\{1-(-1)^{n+1}\right\}$$

This equation immediately shows that

$$I_n = \frac{a^n}{n+1}Q \quad \text{if } n \text{ is even}$$

$$I_n = 0 \quad \text{if } n \text{ is odd}$$

The electrostatic potential at P is therefore equal to

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}} I_n P_n(\cos\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 + \frac{1}{3}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{5}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) + \ldots\right)$$