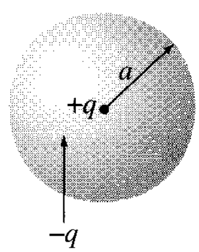
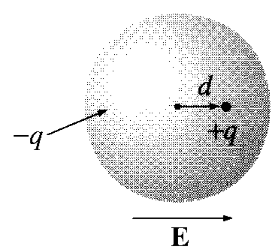
Chapter 4. Electrostatic Fields in Matter

4.1. Polarization

A neutral atom, placed in an external electric field, will experience no net force. However, even though the atom as a whole is neutral, the positive charge is concentrated in the nucleus (radius = 10^{-14} m) while the negative charge forms an electron cloud (radius = 10^{-10} m) surrounding the nucleus.

The nucleus of the atom will experience a force pointing in the same direction as the external electric field of magnitude qE_{ext}. The negatively charged electron cloud will experience a force of the same magnitude, but pointed in a direction opposite to the direction of the electric field. As a result of the external force, the nucleus will move in the direction of the electric field until the external force on it is canceled by the force exerted on the nucleus by the electron cloud.





This leaves the atom **polarized** with plus charge shifted slightly one way and the negative electron cloud in the other direction. A **dipole moment** is created in the atom in the direction of the applied electric field:

$$\vec{p} = \alpha\vec{E}$$

The constant of proportionality α is called the **atomic polarizability**. Its value depends on the detailed structure of the atom.

H	He	Li	Be	C	Ne	Na	Ar	K	Cs
0.667	0.205	24.3	5.60	1.76	0.396	24.1	1.64	43.4	59.6

Table 4.1 Atomic Polarizabilities ($\alpha/4\pi\epsilon_0$, in units of 10^{-30} m^3).

Consider an electron cloud with a constant volume charge density ρ and a radius a. If the total charge of the electron cloud is $-q$ then the corresponding charge density ρ is:

$$\rho = -\frac{q}{\frac{4}{3}\pi a^3} = -\frac{3q}{4\pi a^3}$$

The electric field inside the uniformly charged cloud is:

$$E = \frac{1}{4\pi\varepsilon_o}\frac{qd}{a^3}$$

$$p = qd = (4\pi\varepsilon_o a^3)E = \alpha E \text{ , where } \alpha = 4\pi\varepsilon_o a^3 = 3\varepsilon_o V$$

where V is the volume of the atom, and α is called the **atomic polarizability.**

The magnitude of the induced dipole moment is proportional to the magnitude of the external electric field, and its direction is equal to the direction of the external electric field.

Although this model of the atom is extremely crude, it produces results that are in reasonable agreement with direct measurements of the atomic polarizability.

Example 1:
According to quantum mechanics, the electron cloud for a hydrogen atom in its ground state has a charge density equal to

$$\rho(r) = \frac{q}{\pi a^3}e^{-2r/a}$$

Where q is the charge of the electron and a is the Bohr radius. Find the atomic polarizability of such an atom.

Solution:
As a result of an external electric field the nucleus of the atom will be displaced by a distance d with respect to the center of the electron cloud.

The electric field generated by the electron cloud can be calculated using Gauss's law:

$$E_{cloud} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{d^2} = \frac{1}{4\pi\varepsilon_o}\frac{1}{d^2}\int \rho(r)d\tau = \frac{1}{4\pi\varepsilon_o d^2}\int_0^d \frac{q}{\pi a^3}e^{-2r/a}4\pi r^2 dr$$

$$E_{cloud} = \frac{q}{\pi\varepsilon_o d^2 a^3}\int_0^d e^{-2r/a}r^2 dr$$

Using integration by parts, we get

$$E_{cloud} = \frac{1}{4\pi\varepsilon_o}\frac{q}{d^2}\left[1 - e^{-\frac{2d}{a}}\left(1 + \frac{2d}{a} + \frac{2d^2}{a^2}\right)\right]$$

The displacement of the nucleus will be very small compared to the size of the electron cloud ($d \ll a$). Therefore, we can expand $e^{-\frac{2d}{a}}$ in terms of d/a:

$$e^{-x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$$

$$E_{cloud} = \frac{1}{4\pi\varepsilon_o}\frac{q}{d^2}\left[1 - \left(1 - 2\frac{d}{a} + 2\left(\frac{d}{a}\right)^2 - \frac{4}{3}\left(\frac{d}{a}\right)^3 + \cdots\right)\left(1 + \frac{2d}{a} + \frac{2d^2}{a^2}\right)\right]$$

$$E_{cloud} \cong \frac{1}{4\pi\varepsilon_o}\frac{q}{d^2}\left[\frac{4}{3}\frac{d^3}{a^3}\right] = \frac{1}{3\pi\varepsilon_o}\frac{qd}{a^3} = \frac{1}{3\pi\varepsilon_o}\frac{p}{a^3}$$

The nucleus will be in an equilibrium position when the electric force exerted on it by the external field is equal to the electric force exerted on it by the electron cloud. This occurs when the electric field at the position of the nucleus, generated by the electron cloud, is equal in magnitude to the externally applied electric field, but pointing in the opposite direction. The dipole moment of the dipole can therefore be expressed in terms of the external field:

$$p = 3\pi\varepsilon_o a^3\, E_{ext}$$

The electric polarizability of the material is therefore equal to

$$\alpha = 3\pi\varepsilon_o a^3$$

Which is close to the result obtained using the classical model of the atom.

Plarizability for Molecules:

For molecules the polarizability is different than in atoms because molecules polarize more readily in one direction than the other direction. For example CO_2 molecule is a linear molecule and if electric field is applied along its axis then its polarizability if $4.5 \times 10^{-40}\ C^2.m/N$ but if electric field is applied perpendicular to its axis then its polarizability is just $2 \times 10^{-40}\ C^2.m/N$.

When electric field is at some angle to the axis, then:

$$\vec{p} = \alpha_\perp \vec{E}_\perp + \alpha_\parallel \vec{E}_\parallel$$

In this case the induced dipole moment may not be in the same direction as the applied electric field. For completely asymmetric molecule, the dipole moment can be written as:

$$p_x = \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z$$

$$p_y = \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z$$

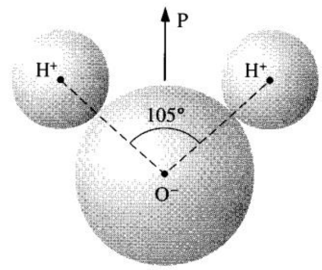
$$p_z = \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z$$

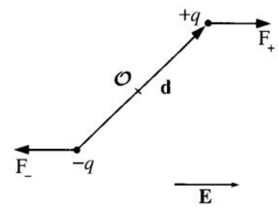
The set of nine constants constitute the polarizability tensor for the molecule. Their values depend on the orientation of the axes and it is possible to choose principal axis such a way to have all the off diagonal elements ($\alpha_{xy}, \alpha_{xz}, \alpha_{yz}$, ...) equal zero, so there are only three polarizability elements α_{xx}, α_{yy}, and α_{zz}.

Alignment of Polar Molecules

Some molecules, like water, have a permanent dipole moment. Normally, the dipole moments of the water molecules will be directed randomly, and the average dipole moment is zero. When the water is exposed to an external electric field, a torque is exerted on the water molecule, and it will try to align its dipole moment with the external electric field. Figure below shows a dipole $\vec{p} = q\vec{d}$ placed in an electric field, directed along the *x*-axis.

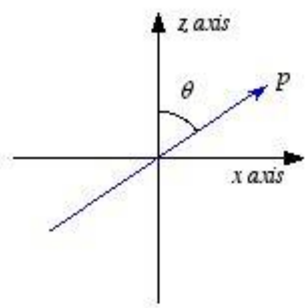
The net force on the dipole is zero since the net charge is equal to zero. The torque on the dipole with respect to its center is equal to





$$\vec{\tau} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$$
$$= \left(\frac{1}{2}\vec{d} \times q\vec{E}\right)$$
$$+ \left(-\frac{1}{2}\vec{d} \times (-q)\vec{E}\right)$$
$$= q\vec{d} \times \vec{E} = \vec{p} \times \vec{E}$$

As a result of this torque, the dipole will try to align itself with the electric field. When the dipole moment is pointing in the same direction as the electric field the torque on the dipole will be equal to zero.



Example 2:
Show that the energy of a dipole in an electric field is given by

$$U = \vec{p}.\vec{E}$$

Consider the dipole located at the origin and making an angle θ with the z-axis of the coordinate system. The electric field is directed along z-axis.

The energy of the system can be determined by calculating the work to be done to move the dipole from infinity to its present location.

Assume the dipole is initially oriented parallel to the *x* axis and is first moved from infinity along the *x* axis to *r* = 0. The force exerted on the dipole by the electric field is directed perpendicular to the displacement and therefore the work done by this force is equal to zero.

The dipole is then rotated to its final position (from $\pi/2$ to θ). The torque exerted by the electric field on the dipole:

$$\vec{\tau} = \vec{p} \times \vec{E} = pE\sin\theta\ \hat{k}$$

In order to rotate the dipole by an external agent, we must supply a torque opposite to $\vec{\tau}$:

$$\vec{\tau}_{applied} = -pE\sin\theta\ \hat{k}$$

Therefore, the work done by an external agent is:

$$W = -\int_{\frac{\pi}{2}}^{\theta} \tau\, d\theta = \int_{\frac{\pi}{2}}^{\theta} pE\sin\theta\ d\theta = -pE\left(\cos\theta - \cos\left(\frac{\pi}{2}\right)\right) = -\vec{p}.\vec{E}$$

The potential energy of the dipole is therefore equal to: $U = -\vec{p}.\vec{E}$
Potential energy reaches a minimum when $\vec{p}$ is parallel to $\vec{E}$ (the dipole is aligned with the electric field).

4.1.4 Polarization

What happens to a piece of dielectric material when placed inside an external electric field?

Well, if the material is made of neutral atoms then field will induce tiny dipole moments in the atoms of the material, pointing in the same direction as the electric field. If the material is made up of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction.

So in both cases material will be **polarized**, and the **polarization** of the material is:

$$\vec{P} = dipole\ moment\ per\ unit\ volume$$

Now this polarized material will produce its own electric field.

4.2. The Field of a Polarized Object

Consider a piece of polarized material with a dipole moment per unit volume equal to $\vec{P}$. Since dipole material is made of several tiny dipoles and the potential due to a single dipole is:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}.\Delta\hat{r}}{(\Delta r)^2}$$

The electrostatic potential generated by the whole material will be integral of the above relation:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int\frac{\Delta\hat{r}.\vec{P}(\vec{r}')}{(\Delta r)^2}d\tau' = \frac{1}{4\pi\varepsilon_o}\int\vec{P}.\vec{\nabla}'\left(\frac{1}{\Delta r}\right)d\tau'$$

where $\Delta\vec{r} = \vec{r} - \vec{r}'$ and $\vec{\nabla}'\left(\frac{1}{\Delta r}\right) = \frac{\Delta\hat{r}}{(\Delta r)^2}$

Using the following relation (one of the product rules of the vector operator)

$$\vec{\nabla}'.\left(\frac{1}{\Delta r}\vec{P}\right) = \frac{1}{\Delta r}(\vec{\nabla}'.\vec{P}) + \vec{P}.\vec{\nabla}\left(\frac{1}{\Delta r}\right)$$

We can rewrite the expression for the electric potential as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int\vec{P}.\vec{\nabla}'(1/\Delta r)d\tau' = \frac{1}{4\pi\varepsilon_o}\left[\int\vec{\nabla}'.\left(\frac{1}{\Delta r}\vec{P}\right)d\tau' - \int\frac{1}{\Delta r}(\vec{\nabla}'.\vec{P})d\tau'\right]$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\left[\int\frac{1}{\Delta r}\vec{P}.d\vec{a}' - \int\frac{1}{\Delta r}(\vec{\nabla}'.\vec{P})d\tau'\right]$$

The first term looks like the potential of a surface charge distribution with:

$\sigma_b = \vec{P}.\hat{n}$ (bound surface charge) where $\hat{n}$ is the normal unit vector

and the second term looks like the potential of a volume charge with:

$\rho_b = -(\vec{\nabla}.\vec{P})$ (bound volume charge)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\int\frac{1}{\Delta r}\sigma_b d\vec{a} + \frac{1}{4\pi\varepsilon_o}\int\frac{1}{\Delta r}\rho_b d\tau$$

The equation for the electrostatic potential shows that the potential (and therefore also the electric field) generated by a polarized object is equal to the potential generated by an object with surface charge density σ_b and volume charge density ρ_b.

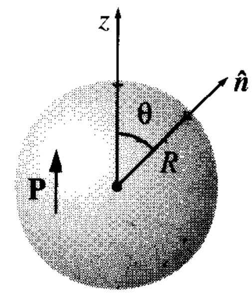
So instead of integrating the contributions of all infinitesimal dipoles, we can just find those bound charges and calculate the potential or field they produce.

Example 3:

Find the electric field produced by a uniformly polarized sphere of radius R.

Solution:

We can choose to have the direction of polarization along z-axis. The volume bound charge density ρ_b is zero since the polarization is uniform:



$$\rho_b = -(\vec{\nabla}.\vec{P}) = 0$$

However, the surface charge density is:

$$\sigma_b = \vec{P}.\hat{n} = P\cos\theta$$

Where θ is the usual polar coordinate.

We would like to find the electric field produced by this surface charge density "pasted" on the sphere.

We already have calculated electric potential due to a sphere on which charge density $\sigma_b = k\cos\theta$ was pasted and we found that:

The potential inside the sphere is therefore

$$V(r,\theta) = \frac{k}{3\epsilon_0} r\cos\theta \qquad (r \le R),$$

whereas outside the sphere

$$V(r,\theta) = \frac{kR^3}{3\epsilon_0}\frac{1}{r^2}\cos\theta \qquad (r \ge R).$$

So the electric potential due to $\sigma_b = P\cos\theta$ is given as:

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_o} r\cos\theta & for\ r \le R \\ \dfrac{P}{3\epsilon_o}\dfrac{R^3}{r^2}\cos\theta & for\ r \ge R \end{cases}$$

Since $r\cos\theta = z$

So the electric field inside the sphere $(r \le R)$ is:

$$\vec{E}_{in} = -\vec{\nabla} V = -\left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial x}\hat{\jmath} + \frac{\partial}{\partial x}\hat{k}\right)\frac{P}{3\epsilon_o}\, z = -\frac{P}{3\epsilon_o}\hat{k}$$

Outside the sphere, the potential is identical to that of a perfect dipole at the origin.

$$V = \frac{1}{4\pi\epsilon_o}\frac{\vec{p}.\hat{r}}{r^2} \qquad for\ r \geq R$$

Whose dipole moment is equal to the dipole moment of a uniformly polarized sphere.

$$\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$$

$$V = \frac{1}{4\pi\epsilon_o}\frac{\frac{4}{3}\pi R^3}{r^2} P\cos\theta = \frac{P}{3\epsilon_o}\frac{R^3}{r^2}\cos\theta \qquad for\ r \geq R$$

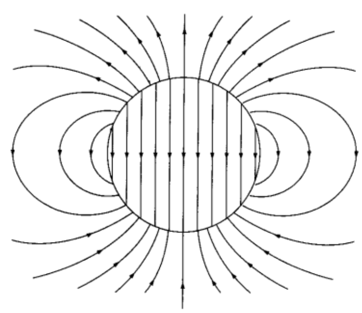
$$\boxed{\mathbf{E}_{\text{dip}}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).}$$

The electric field outside the sphere is:

$$\vec{E}_{out} = -\vec{\nabla} V = -\left(\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}\right)\frac{P}{3\epsilon_o}\frac{R^3}{r^2}\cos\theta$$

$$\vec{E}_{out} = -\left(-2\frac{P}{3\epsilon_o}\frac{R^3}{r^3}\cos\theta\,\hat{r} - \frac{P}{3\epsilon_o}\frac{R^3}{r^3}\sin\theta\,\hat{\theta} + 0\right)$$

$$\vec{E}_{out} = \frac{P}{3\epsilon_o}\frac{R^3}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



Electric field inside and outside the uniformly polarized sphere is shown in the figure.

Example 4:

A sphere of radius R carries a polarization $\vec{P} = k\vec{r}$, where k is a constant and $\vec{r}$ is the vector from the center.
a) Calculate the bound charges σ_b and ρ_b.
b) Find the electric field inside and outside the sphere.

a) The unit vector $\hat{n}$ on the surface of the sphere is equal to the radial unit vector. The bound surface charge is:

$$\sigma_b = \vec{P}.\hat{n}\Big|_{r=R} = k\vec{r}.\hat{r}|_{r=R} = kR$$

The bound volume charge density is:

$$\rho_b = -\left(\vec{\nabla}.\vec{P}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) = -3k$$

b) First consider the region outside the sphere. The electric field in this region due to the surface charge and due to the volume charge.

Electric field outside the sphere due to surface charge is:

$$\vec{E}_{out-surface} = \frac{1}{4\pi\varepsilon_o}\frac{1}{r^2}\int \sigma_b da'\,\hat{r} = \frac{1}{4\pi\varepsilon_o}\frac{kR}{r^2} * 4\pi R^2 = \frac{kR^3}{\varepsilon_o r^2}\hat{r}$$

The electric field in this region due to the volume charge is:

$$\vec{E}_{out-volume}(r) = \frac{1}{4\pi\varepsilon_o}\frac{1}{r^2}\int \rho_b d\tau\,\hat{r} = \frac{1}{4\pi\varepsilon_o}\frac{(-3k)}{r^2} * \frac{4}{3}\pi R^3\hat{r} = -\frac{kR^3}{\varepsilon_o r^2}\hat{r}$$

$$\vec{E}_{out}(r) = \vec{E}_{surface}(r) + \vec{E}_{volume}(r) = 0$$

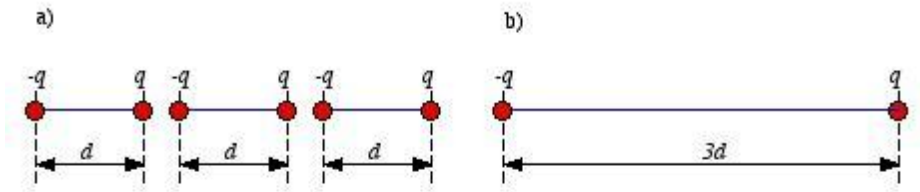
Now consider the region inside the sphere. The electric field in this region due to the surface charge is equal to zero.

The electric field due to the volume charge is equal to

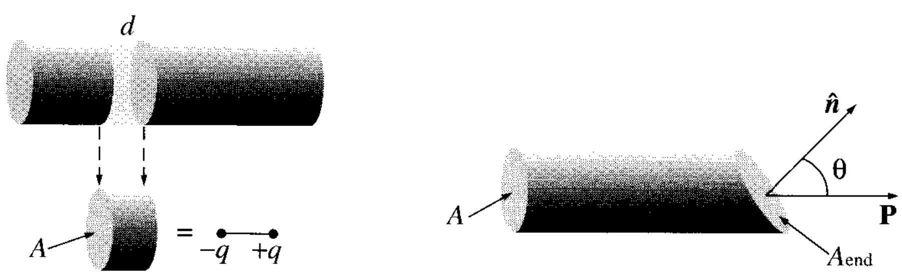
$$\vec{E}_{in-volume}(r) = \frac{1}{4\pi\varepsilon_o}\frac{1}{r^2}\int \rho_b d\tau\,\hat{r} = \frac{1}{4\pi\varepsilon_o}\frac{1}{r^2}\,(-3k)\frac{4}{3}\pi r^3 = -\frac{kr}{\varepsilon_o}\hat{r}$$

Physical Interpretation of Bound Charges:

The bound charges introduced in this Section are not just mathematical artifacts, but are real charges, bound to the individual dipoles of the material. Consider for example the three dipoles shown in the figure below. When they are aligned (lengthwise) the center charges cancel, and the system looks like a single dipole with dipole moment $3dq$.



To calculate the actual amount of bound charge resulting from a given polarization, lets consider a tube of dielectric parallel to polarization $\vec{P}$. The dipole moment of the tiny chunk shown in the figure below is P(Ad), where A is the cross-sectional area and d is the thickness of the tube.



In terms of charge q at the ends, the same dipole moment can be written as qd.

$$qd = P(Ad)$$

$$q = PA$$

If the ends are sliced off perpendicularly then

$$\sigma_b = \frac{q}{A} = P$$

But if it is sliced at an oblique angle as shown then

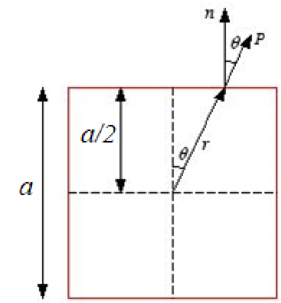
$$\sigma_b = P\cos\theta = \vec{P}.\hat{n}$$

The effect of polarization is to paint a bound charge $\sigma_b = \vec{P}.\hat{n}$ over the surface of the material.

Since these charges reside on the surface and are bound to the dipoles they are called the **bound surface charge** (σ_b).

Example 5:

A dielectric cube of side *s*, centered at the origin, carries a "frozen-in" polarization $\vec{P} = k\vec{r}$, where *k* is a constant. Find all the bound charges, and check that they add up to zero.



The bound volume charge density ρ_b is equal to:

$$\rho_b = -[\vec{\nabla}.\vec{P}] = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) = -3k$$

Since the bound volume charge density is constant, the total bound volume charge in the cube is equal to product of the charge density and the volume:

$$q_{volume} = -3ka^3$$

The surface charge density σ_b is equal to:

$$\sigma_b = \vec{P}.\hat{n} = k\vec{r}.\hat{n}$$

The scalar product between $\vec{r}$ and $\hat{n}$ can be evaluate as:

$$\vec{r}.\hat{n} = r\cos\theta = \frac{a}{2}$$

Therefore the surface charge density is equal to

$$\sigma_b = k\vec{r}.\hat{n} = \frac{ka}{2}$$

$$q_{surface} = \frac{ka}{2}(6a^2) = 3ka^3$$

The total bound charge on the cube is equal to

$$q_{total} = q_{volume} + q_{surface} = 0$$

4.3. The Electric Displacement

The electric field generated by a polarized material is equal to the electric field produced by its bound charges. If free charges are also present then the total electric field produced by this system is equal to the vector sum of the electric fields produced by the bound charges and by the free charges.

Gauss's law can also be used for this type of systems to calculate the electric field as long as we include both free and bound charges:

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_o} = \frac{\rho_{bound} + \rho_{free}}{\varepsilon_o} = \frac{1}{\varepsilon_o}(-\vec{\nabla}.\vec{P} + \rho_{free})$$

where $\vec{P}$ is the polarization of the material. This expression can be rewritten as:

$$\vec{\nabla}.(\varepsilon_o\vec{E} + \vec{P}) = \vec{\nabla}.\vec{D} = \rho_{free}$$

Where $\vec{D}(= \varepsilon_o\vec{E} + \vec{P})$ is called the **electric displacement**.

Gauss's law can also be rewritten as:

$$\vec{\nabla}.\vec{D} = \rho_{free} \qquad \text{(Gauss's law in differential form)}$$

and

$$\oint \vec{D}.d\vec{a} = Q_{free} \qquad \text{(Gauss's law in integral form)}$$

These two versions of Gauss's law are particularly useful since they make reference only to free charges, which are the charges we can control.
Although it seems that the displacement $\vec{D}$ has properties similar to the electric field $\vec{E}$ there are some very significant differences. For example, the curl of $\vec{D}$ is equal to:

$$\vec{\nabla} \times \vec{D} = \varepsilon_o \vec{\nabla} \times \vec{E} + \vec{\nabla} \times \vec{P} = \vec{\nabla} \times \vec{P}$$

and is in general not equal to zero.

Since curl of $\vec{D}$ is not necessarily equal to zero, there is in general no potential that generates $\vec{D}$.

The Helmholtz theorem tell us that if we know the curl and the divergence of a vector function $\vec{v}$ then this is sufficient information to uniquely define the vector function $\vec{v}$.

Therefore, the electric field $\vec{E}$ is uniquely defined by Gauss's law since we know that he curl of $\vec{E}$ is zero, everywhere. The displacement vector $\vec{D}$ on the other hand is not uniquely determined by the free charge distribution, but requires additional information (such as $\vec{P}$).

Boundary Conditions:

The electrostatic boundary conditions can be recast interms of $\vec{D}$, such that:

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

Since $\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$, so we can write the other boundary condition as:

$$D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$

In the presence of a dielectric these boundary conditions are more useful than electric field boundary conditions:

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_o}$$

$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

Example 6:
Suppose the field inside a large piece of dielectric is $\vec{E}_o$, so that the electric displacement is equal to $\vec{D}_o = \varepsilon_o \vec{E}_o + \vec{P}$.
a) Now, a small spherical cavity is hollowed out of the material. Find the field at the center of the cavity in terms of $\vec{E}_o$ and $\vec{P}$. Also find the displacement at the center of the cavity in terms of $\vec{D}_o$ and $\vec{P}$.
b) Do the same for a long needle-shaped cavity running parallel to $\vec{P}$.
c) Do the same for a thin wafer-shaped cavity perpendicular to $\vec{P}$.

a) Consider a large piece of dielectric material with polarization $\vec{P}$ and a small sphere with polarization $-\vec{P}$ superimposed on it. The field generated by this system is equal to the field generated by the dielectric material with a small spherical cavity hollowed out (principle of superposition).

The electric field inside a sphere with polarization $-\vec{P}$ is uniform and equal to [as calculated in example 3]

$$\vec{E}_{sphere} = -\frac{1}{3\varepsilon_o}\left(-\vec{P}\right) = \frac{\vec{P}}{3\varepsilon_o}$$

The field at the center of the cavity is therefore equal to

$$\vec{E}_{center} = \vec{E}_o + \vec{E}_{sphere} = \vec{E}_o + \frac{\vec{P}}{3\varepsilon_o}$$

The corresponding electric displacement at the center of the cavity is equal to

$$\vec{D}_{center} = \varepsilon_o \vec{E}_{center} = \varepsilon_o \vec{E}_o + \frac{1}{3}\vec{P} = \vec{D}_o - \frac{2}{3}\vec{P}$$

b) Consider a large piece of dielectric material with polarization $\vec{P}$ and a small long needle-shaped piece with polarization $-\vec{P}$ superimposed on it. The field generated by this system is equal to the field generated by the dielectric material with a small long needle-shaped cavity hollowed out (principle of superposition). The electric field of a polarized needle of length *s* is equal to that of two point charges (+*q* and -*q*) located a distance *s* apart. The charge on top of the needle will be negative, while the charge on the bottom of the needle will be positive. The charge density on the end caps of the needle is equal to *P*. Therefore,

$$q = \sigma_b A = PA$$

where *A* is the surface area of the end caps of the needle. If s is the total length of the needle then the electric field generated by the needle at its center is:

$$\vec{E}_{needle} = \frac{1}{4\pi\varepsilon_o}\frac{+PA}{s^2/4}\,\hat{k} - \frac{1}{4\pi\varepsilon_o}\frac{(-PA)}{\frac{s^2}{4}}\,\hat{k} = \frac{2}{\pi\varepsilon_o}\frac{PA}{s^2}\,\hat{k}$$

In the needle limit $A \to 0$ and therefore $\vec{E}_{needle} \to 0$. Thus at the center of the needle cavity

$$\vec{E}_{center} = \vec{E}_o$$

The electric displacement at this point is equal to:

$$\vec{D}_{center} = \varepsilon_o \vec{E}_o = \vec{D}_o - \vec{P}$$

c) Consider a large piece of dielectric material with polarization $\vec{P}$ and a thin wafer-shaped piece of dielectric material with polarization $-\vec{P}$ superimposed on it. The field generated by this system is equal to the field generated by the dielectric material with a thin wafer-shaped cavity hollowed out (principle of superposition).

The electric field inside the wafer will be that of two parallel plates with charge densities equal to $-\sigma$ on the top and $+\sigma$ on the bottom. For a thin wafer-shaped cavity the electric field between the plates will be equal to the field of a parallel-plate capacitor with infinitely large plates. Thus

$$\vec{E}_{wafer} = \frac{\sigma}{\varepsilon_o}\hat{k} = \frac{1}{\varepsilon_o}\vec{P}$$

The net electric field in the center of the cavity is therefore equal to

$$\vec{E}_{center} = \vec{E}_o + \vec{E}_{wafer} = \vec{E}_o + \frac{1}{\varepsilon_o}\vec{P}$$

The electric displacement at the center of the cavity is equal to

$$\vec{D}_{center} = \varepsilon_o \vec{E}_{center} = \varepsilon_o \vec{E}_o + \vec{P} = \vec{D}$$

4.4. Linear Dielectrics

Most dielectric materials become polarized when they are placed in an external electric field. In many materials the polarization is proportional to the electric field:

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

where $\vec{E}$ is the **total** electric field (external + internal). The constant of proportionality χ_e is called the **electric susceptibility**.

Materials in which the induced polarization is proportional to the electric field are called **linear dielectrics**.

The electric displacement in a linear dielectric is also proportional to the total electric field:

$$\vec{D} = \varepsilon_o \vec{E}_o + \vec{P} = \varepsilon_o (1 + \chi_e)\vec{E} = \varepsilon \vec{E}$$

where ε is called the **permittivity** of the material which is equal to

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_o} = (1 + \chi_e)$$

The constant ε_r is called **relative permittivity** or **dielectric constant** of the material.

Consider a volume *V* entirely filled with linear dielectric material with dielectric constant ε_r. The polarization $\vec{P}$ of this material is:

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

and is therefore proportional to $\vec{E}$ everywhere. Therefore:

$$\vec{\nabla} \times \vec{P} = \varepsilon_o \chi_e \left(\vec{\nabla} \times \vec{E}\right) = 0$$

and consequently

$$\vec{\nabla} \times \vec{D} = \varepsilon_o \left(\vec{\nabla} \times \vec{E}\right) + \vec{\nabla} \times \vec{P} = 0$$

The electric displacement $\vec{D}$ therefore satisfies the following two conditions:

$$\vec{\nabla} \times \vec{D} = 0 \text{ and } \vec{\nabla}.\vec{D} = \rho_{free}$$

The electric field generated by the free charges when the dielectric is not present satisfies the following two equations:

$$\vec{\nabla} \times \vec{E}_{free} = 0 \text{ and } \vec{\nabla}.\vec{E}_{free} = \frac{\rho_{free}}{\varepsilon_o}$$

Comparing the two sets of differential equations for $\vec{D}$ and $\vec{E}_{free}$ we conclude that:

$$\vec{D} = \varepsilon_o \vec{E}_{free}$$

The displacement $\vec{D}$ can also be expressed in terms of the total field inside the dielectric:

$$\vec{D} = \varepsilon_o (1 + \chi_e) \vec{E} = \varepsilon \vec{E}$$

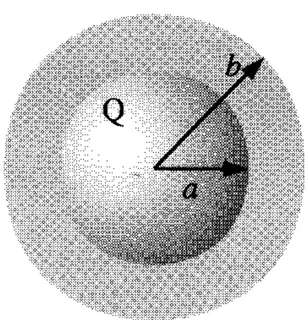
These two equations show that

$$\vec{E} = \frac{\varepsilon_o}{\varepsilon} \vec{E}_{free} = \frac{1}{\varepsilon_r} \vec{E}_{free}$$

The presence of the dielectric material therefore reduces the electric field by a factor ε_r.

Example 7:

A metal sphere of radius a carries a carries a charge Q, it is surrounded out to radius b, by a linear dielectric material of permittivity ε, Find the potential at the center (relative to infinity)



Solution:

To compute the potential we need to know the electric field everywhere and to find the electric field let's calculate displacement vector D:

For a linear dielectric material:

$$\vec{\nabla}.\vec{D} = \rho_{free}$$

Or using Gauss's theorem: $\int \vec{\nabla}.\vec{D}\, d\tau = \int \vec{D}.d\vec{a} = \int \rho_{free} d\tau = Q$

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

And hence electric field will be:

$$\vec{E} = \begin{cases} \frac{1}{4\pi\varepsilon}\frac{Q}{r^2}\hat{r} \quad for \quad b < r < a \\ \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}\hat{r} \quad for \quad r > b \\ 0 \quad for \quad r < a \end{cases}$$

The potential at the center will be:

$$V = -\int_{\infty}^{0} \vec{E}.d\vec{l} = -\int_{\infty}^{b} \frac{Q}{4\pi\varepsilon_o r^2} dr - \int_{b}^{a} \frac{Q}{4\pi\varepsilon r^2} dr + -\int_{a}^{0} 0\, dr$$

$$V = \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_o b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right]$$

We can also find the polarization of the material surrounding the conducting sphere:

$$\vec{P} = \varepsilon_o\chi_e\vec{E} = \frac{\varepsilon_o\chi_e Q}{4\pi\varepsilon r^2}\hat{r} = \frac{\chi_e Q}{4\pi\varepsilon_r r^2}\hat{r}$$

And the bound volume charge distribution would be:

$$\rho_b = -\vec{\nabla}.\vec{P} = -(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\chi_e Q}{4\pi\varepsilon_r r^2}\right) = 0$$

And bound surface charges will be:

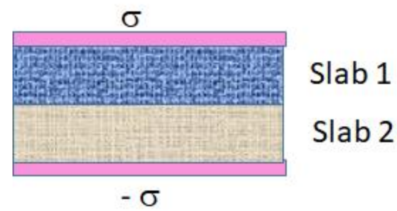
$$\rho_b = \vec{P}.\hat{n} = \begin{cases} \frac{\chi_e Q}{4\pi\varepsilon_r b^2} \quad at\ the\ outer\ surface \\ \frac{-\chi_e Q}{4\pi\varepsilon_r a^2} \quad at\ the\ inner\ surface \end{cases}$$

The surface charge at a is negative because $\hat{n}$ points outwards with respect to dielectric and at a the direction of $\hat{n}$ would be $-\vec{r}$, whereas at b the direction of $\hat{n}$ would be $+\vec{r}$ direction.

And this makes sense because the charge on the conducting sphere is positive and it will attractive negative charge from the dielectric material

Example 7:
The space between the plates of a parallel-plate capacitor is filled with two slabs of linear dielectric material. Each slab has thickness a, so that the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is σ and on the bottom plate is $-\sigma$.

a) Find the electric displacement $\vec{D}$ in each slab.
b) Find the electric field $\vec{E}$ in each slab.
c) Find the polarization $\vec{P}$ in each slab.
d) Find the potential difference between the plates.
e) Find the location and amount of all bound charge.



Solution:

a) The electric displacement $\vec{D}$ in slab 1 can be calculated using "Gauss's law". Consider a cylinder with cross sectional area A and axis parallel to the z-axis, being used as a Gaussian surface. The top of the cylinder is located inside the top metal plate (where the electric displacement is zero) and the bottom of the cylinder is located inside the dielectric of slab 1. The electric displacement is directed parallel to the z-axis and pointed downwards. Therefore, the displacement flux through this surface is equal to

$$\Phi_D = D_1 A = Q_{free,enclosed} = \sigma A$$
$$D_1 = \sigma$$

In vector notation: $\vec{D}_1 = -\sigma\hat{k}$

Similarly: $\vec{D}_2 = -\sigma\hat{k}$

b) The electric field $\vec{E}_1$ in slab 1 is:

$$\vec{D}_1 = \varepsilon\vec{E}_1 = \varepsilon_o\varepsilon_{r_1}\vec{E}_1 = -\sigma\hat{k}$$
$$\vec{E}_1 = -\frac{\sigma}{\varepsilon_o\varepsilon_{r_1}}\hat{k} = -\frac{\sigma}{2\varepsilon_o}\hat{k}$$

The electric field $\vec{E}_2$ in slab 2 is:

$$\vec{E}_2 = -\frac{\sigma}{\varepsilon_o\varepsilon_{r_2}}\hat{k} = -\frac{\sigma}{1.5\varepsilon_o}\hat{k} = -\frac{2\sigma}{3\varepsilon_o}\hat{k}$$

c) The polarization $\vec{P}$ can be calculated:

$$\vec{P} = \vec{D} - \varepsilon_o\vec{E}$$

The polarization of slab 1:

$$\vec{P}_1 = \vec{D}_1 - \varepsilon_o\vec{E}_1 = -\sigma\hat{k} - \varepsilon_o\left(-\frac{\sigma}{2\varepsilon_o}\hat{k}\right) = -\frac{\sigma}{2}\hat{k}$$

The polarization of slab 2:

$$\vec{P}_2 = \vec{D}_2 - \varepsilon_o \vec{E}_2 = -\sigma\hat{k} - \varepsilon_o\left(-\frac{2\sigma}{3\varepsilon_o}\hat{k}\right) = -\frac{\sigma}{3}\hat{k}$$

d) The potential difference between the top plate and the bottom plate can be calculated from the electric field:

$$\Delta V = V_{top} - V_{bottom} = -\int_{bottom}^{top} \vec{E}.d\vec{l} = E_1 a + E_2 a = \left(\frac{\sigma}{2\varepsilon_o} + \frac{2\sigma}{3\varepsilon_o}\right)a = \frac{7\sigma a}{3\varepsilon_o}$$

e) There are no bound volume charges (constant polarization $-\vec{\nabla}.\vec{P} = 0 = \rho_{free}$). The bound surface charge density on top of slab 1 is:

$$\sigma_{b,top,1} = \vec{P}_1.\hat{n} = -\frac{\sigma}{2}\hat{k}.\hat{k} = -\frac{\sigma}{2}$$

The surface charge density on the bottom of slab 1 is equal to

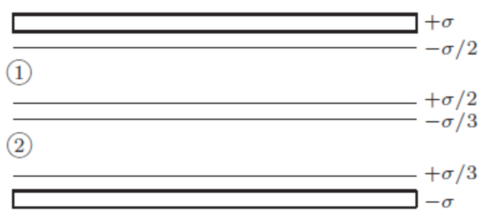
$$\sigma_{b,bottom,1} = \vec{P}_1.\hat{n} = -\frac{\sigma}{2}\hat{k}.(-\hat{k}) = \frac{\sigma}{2}$$

The surface charge density on top of slab 2 is:

$$\sigma_{b,top,2} = \vec{P}_2.\hat{n} = -\frac{\sigma}{3}\hat{k}.\hat{k} = -\frac{\sigma}{3}$$

The surface charge density on the bottom of slab 2:

$$\sigma_{b,bottom,2} = \vec{P}_2.\hat{n} = -\frac{\sigma}{3}\hat{k}.(-\hat{k}) = \frac{\sigma}{3}$$



f) The total charge above slab 1 is equal to σ - σ/2 = σ/2. This charge will produce an electric field in slab 1 equal to

$$\vec{E}_{1,above} = -\frac{\left(\frac{\sigma}{2}\right)}{2\varepsilon_o}\hat{k} = -\frac{\sigma}{4\varepsilon_o}\hat{k}$$

The total charge below slab 1 is equal to σ/2 - σ/3 + σ/3 - σ = - σ/2. This charge will produce an electric field in slab 1 equal to:

$$\vec{E}_{1,below} = -\frac{\sigma}{4\varepsilon_o}\hat{k}$$

The total electric field in slab 1 is the vector sum of these two fields and is equal to

$$\vec{E}_1 = \vec{E}_{1,above} + \vec{E}_{1,below} = -\frac{\sigma}{2\varepsilon_o}\hat{k}$$

The total charge above slab 2 is equal to σ - σ/2 + σ/2 - σ/3 = 2σ/3. This charge will produce an electric field in slab 2 equal to

$$\vec{E}_{2,above} = -\frac{\left(\frac{2\sigma}{3}\right)}{2\varepsilon_o}\hat{k} = -\frac{\sigma}{3\varepsilon_o}\hat{k}$$

The total charge below slab 1 is equal to σ/3 - σ = - 2σ/3. This charge will produce an electric field in slab 1 equal to

$$\vec{E}_{2,below} = -\frac{\sigma}{3\varepsilon_o}\hat{k}$$

The total electric field in slab 1 is the vector sum of these two fields and is equal to

$$\vec{E}_2 = \vec{E}_{2,above} + \vec{E}_{2,below} = -\frac{2\sigma}{3\varepsilon_o}\hat{k}$$

These answers are in agreement with the results obtained in part b).

4.5. Energy in dielectric systems

Consider a capacitor with capacitance C and charged up to a potential V. The total energy stored in the capacitor is equal to the work done during the charging process:

$$W = \frac{1}{2}CV^2$$

If the capacitor is filled with a linear dielectric (dielectric constant ε_r) then the total capacitance will increase by a factor ε_r:

$$C = \varepsilon_r C_{air}$$

consequently the energy stored in the capacitor (when held at a constant potential) is increased by a factor ε_r.

A general expression for the energy of a capacitor with dielectric materials present can be found by studying the charging process in detail. Consider a free charge ρ_{free} held at a potential V. During the charging process the free charge is increased by $\Delta\rho_{free}$. The work done on the extra free charge is equal to:

$$\Delta W = \int \Delta\rho_{free} * V * d\tau$$

Since the divergence of the electric displacement $\vec{D}$ is equal to the free charge density ρ_{free} , the divergence of $\Delta\vec{D}$ is equal to $\Delta\rho_{free}$. Therefore:

$$\Delta W = \int (\vec{\nabla}.\Delta\vec{D})V d\tau$$

Using the following relation:

$$(\vec{\nabla}.V\Delta\vec{D}) = (\vec{\nabla}.\Delta\vec{D})V + (\vec{\nabla}V).\Delta\vec{D}$$

we can rewrite the expression for ΔW as

$$\Delta W = \int \vec{\nabla}.(V\Delta\vec{D})d\tau - \int \left[(\vec{\nabla}V).\Delta\vec{D}\right]d\tau$$

The first term on the right-hand side of this equation can be rewritten as

$$\int \vec{\nabla}.(V\Delta\vec{D})V d\tau = \int (V\Delta\vec{D}).d\vec{a} = 0$$

Since the product of potential and electric displacement approach zero faster than $1/r^2$ when r approached infinity. Therefore,

$$\Delta W = -\int \left[(\vec{\nabla}V).\Delta\vec{D}\right]d\tau = \int (\vec{E}.\Delta\vec{D})d\tau$$

Assuming that the materials present in the system are linear dielectrics then

$$\vec{D} = \varepsilon\vec{E}$$

Hence: $\vec{E}.\Delta\vec{D} = \vec{E}.\varepsilon\Delta\vec{E} = \frac{1}{2}\Delta(\varepsilon\vec{E}.\vec{E}) = \frac{1}{2}\Delta(\vec{D}.\vec{E})$

$$\Delta W = \int \left(\vec{E}.\Delta\vec{D}\right)d\tau = \frac{1}{2}\Delta\int \left(\vec{D}.\vec{E}\right)d\tau$$

The total work done during the charging process is therefore equal to

$$W = \frac{1}{2}\int \left(\vec{D}.\vec{E}\right)d\tau$$

Note: this equation can be used to calculate the energy for a system that contains linear dielectrics. If some materials in the system are non-linear dielectrics then the derivation given above $\left[\vec{E}.\Delta\vec{D} \neq \frac{1}{2}\Delta\left(\vec{D}.\vec{E}\right)\right]$ is not correct for non-linear dielectrics).

Example 8:
A spherical conductor, of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out to a radius b. Find the energy of this configuration.
Since the system has spherical symmetry the electric displacement $\vec{D}$ is completely determined by the free charge.

$$\vec{D}(\vec{r}) = \frac{1}{4\pi r^2}\ Q_{encl} = 0 \qquad\qquad r < a$$

$$\vec{D}(\vec{r}) = \frac{1}{4\pi}\frac{Q}{r^2} = 0 \qquad\qquad r > a$$

Since we are dealing with linear dielectrics, the electric field $\vec{E}$ is equal to $\vec{D}/\left(\varepsilon_o(1+\chi_e)\right)$. Taking into account that the susceptibility of vacuum is zero and the susceptibility of a conductor is infinite we get:

$$\vec{E}(\vec{r}) = 0 \qquad\qquad r < a$$

$$\vec{E}(\vec{r}) = \frac{\vec{D}(\vec{r})}{\varepsilon_o(1+\chi_e)} = \frac{1}{4\pi\varepsilon_o(1+\chi_e)}\frac{Q}{r^2} = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2} \qquad\qquad a < r < b$$

$$\vec{E}(\vec{r}) = \frac{\vec{D}(\vec{r})}{\varepsilon_o} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2} \qquad\qquad r > b$$

The scalar product $\vec{D}.\vec{E}$ is equal to $\left|\vec{D}\right|\left|\vec{E}\right|$ since $\vec{E}$ and $\vec{D}$ are parallel, everywhere. The energy of the system is:

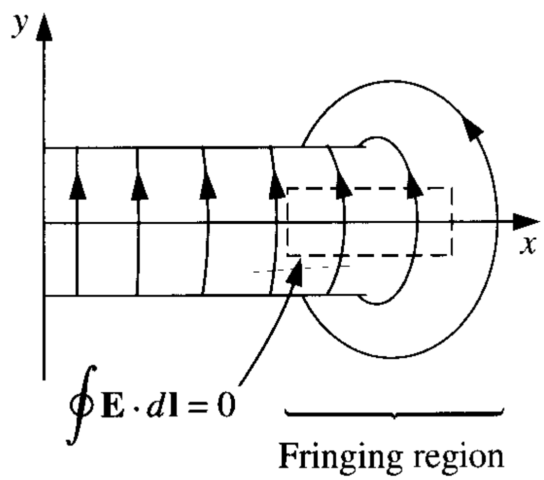
$$W = \frac{1}{2}\int \left(\vec{D}.\vec{E}\right)d\tau = 2\pi\int_a^\infty \left|\vec{D}\right|\left|\vec{E}\right| r^2 d\tau$$

$$W = 2\pi\int_a^b \frac{1}{16\pi^2\varepsilon_o(1+\chi_e)}\ \frac{Q^2}{r^4}\ r^2 dr + 2\pi\int_b^\infty \frac{1}{16\pi^2\varepsilon_o}\ \frac{Q^2}{r^4}\ r^2 dr$$

$$W = \frac{Q^2}{8\pi\varepsilon_o}\left[\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right] = \frac{Q^2}{8\pi\varepsilon_o(1+\chi_e)}\left[\frac{1}{a}+\frac{\chi_e}{b}\right]$$

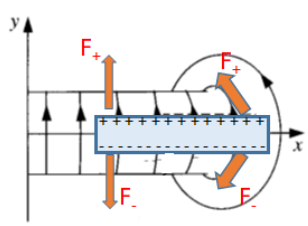
4.6. Forces on dielectrics

A dielectric slab placed partly between the plates of a parallel-plate capacitor will be pulled inside the capacitor. This force is a result of the fringing fields around the edges of the parallel-plate capacitor. **Note: the field outside the capacitor can not be zero** since otherwise the line integral of the electric field around a closed loop, partly inside the capacitor and partly outside the capacitor, would not be equal to zero.



Inside the capacitor, the electric field is uniform. The electric force exerted by the field on the positive bound charge of the dielectric is directed upwards and is canceled by the electric force on the negative bound charge directed downwards.

Outside the capacitor however, the electric field is not uniform and the electric force acting on the positive bound charge will not be canceled by the electric force acting on the negative bound charge. For the system shown in the figure the vertical components of the two forces (outside the capacitor) will cancel, but the horizontal components are pointing in the same direction and therefore do not cancel. The result is a net force acting on the slab, directed towards the center of the capacitor.



A direct calculation of this force requires a knowledge of the fringing fields of the capacitor which are often not well known and difficult to calculate. An alternative method that can be used to determine this force is to calculate the change in the energy of the system when the dielectric is displaced by a distance *ds*. The work to be done to pull the dielectric out by an infinitesimal distance *ds* is equal to

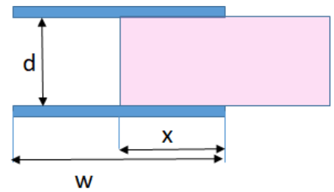
$$dW = F_{ext} dx$$

where F_{ext} is the force provided by the external agent to pull the slab out of the capacitor.

This force must be equal and opposite to the force applied by the field:

$$F_{field} = -F_{ext} = -\frac{dW}{dx}$$

The work done by an external agent to move the slab must be equal to the change in the energy of the capacitor (conservation of energy). Consider the situation shown in the figure below where the slab of dielectric is inserted to a depth x in the capacitor.



The capacitance of this system is:

$$C = C_{vac} + C_{dielectric} = \frac{\varepsilon_o (w-x)a}{d} + \varepsilon_r \frac{\varepsilon_o x a}{d}$$

$$C = \frac{\varepsilon_o x a}{d}(w + \chi_e x)$$

If the total charge on the top plate is Q then the energy stored in the capacitor is equal to

$$W = \frac{Q^2}{2C} = \frac{Q^2}{2} \frac{d}{\varepsilon_o a (w + \chi_e x)}$$

The force on the dielectric can now be calculated and is equal to

$$F_{field} = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

Example 9:

Two long coaxial cylindrical metal tubes (inner radius *a*, outer radius *b*) stand vertically in a tank of dielectric oil (susceptibility χ_e, mass density ρ). The inner one is maintained at potential *V*, and the outer one is grounded. To what height *x* does the oil rise in the space between the tubes?

The height of the oil is such that the electric force on the oil balances the gravitational force. The capacitance of an empty cylindrical capacitor of height *H* is equal to

$$C = \frac{2\pi\varepsilon_o H}{\ln\left(\frac{b}{a}\right)}$$

If the oil rises to a height *h* then the capacitance of the capacitor is equal to

$$C = C_{vac} + C_{dielctric} = \frac{2\pi\varepsilon_o(H - x)}{\ln\left(\frac{b}{a}\right)} + (1 + \chi_e)\frac{2\pi\varepsilon_o x}{\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon_o}{\ln\left(\frac{b}{a}\right)}(H + x\chi_e)$$

The electric force on the dielectric (the oil) is [directed upwards]:

$$F_{field} = \frac{1}{2}V^2\frac{dC}{dx} = \frac{1}{2}V^2 * \frac{2\pi\varepsilon_o}{\ln\left(\frac{b}{a}\right)} * \chi_e = \chi_e\frac{\pi\varepsilon_o}{\ln\left(\frac{b}{a}\right)}V^2$$

The gravitational force acting on the oil is [directed downwards]:

$$F_g = \pi(b^2 - a^2)h\rho g$$

In the equilibrium position: $|F_{grav}| = |F_{el}|$. Thus

$$F_{field} = F_g$$

$$\chi_e\frac{\pi\varepsilon_o}{\ln\left(\frac{b}{a}\right)}V^2 = \pi(b^2 - a^2)h\rho g$$

$$h = \chi_e\frac{\varepsilon_o}{\rho g(b^2 - a^2)\ln\left(\frac{b}{a}\right)}V^2$$