Chapter 6. Magnetostatic Fields in Matter

6.1.1 Magnetization

Any macroscopic object consists of many atoms or molecules, each having electric charges in motion. With each electron in an atom or molecule we can associate a tiny magnetic dipole moment (due to its spin). Ordinarily, the individual dipoles cancel each other because of the random orientation of their direction. However, when a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and the material becomes magnetized. The state of magnetic polarization of a material is described by the parameter M which is called the **magnetization** of the material and is defined as

M=magnetic dipole moment per unit volume

In some material the magnetization is parallel to $\vec{B}$. These materials are called **paramagnetic.**

In other materials the magnetization is opposite to $\vec{B}$. These materials are called **diamagnetic.**

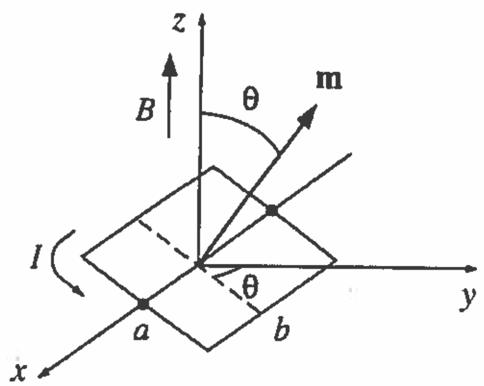
A third group of materials, also called **ferromagnetic** materials, retain a substantial magnetization indefinitely after the external field has been removed.

6.1.2. Torque and Forces on Magnetic Dipole

A magnetic dipole experience a torque in a magnetic field as electric dipole experiences a torque in an electric field.

Lets consider a current loop tilted at an angle θ from the z-axis as shown in the figure below and $\vec{B}$ pointing in the z-direction.

The force on the right sloping side is equal and opposite to the force on the left sloping side and they cancel each other and just stretch the loop in that direction. Whereas force on the horizontal portions of the loop (top and bottom potions) also cancel each other but do create a torque.



$$\vec{N} = aF \sin\theta \ \hat{\imath}$$

The magnitude of the force on each segment is:

$$F = ibB$$

$$\vec{N} = iabB \sin\theta \, \hat{\imath}$$

Where ab is the area of the loop:

$$\vec{N} = i\vec{A} \times \vec{B} = \vec{m} \times \vec{B}$$

Where m is the magnetic dipole moment $m = iab$

The torque is in such a direction that it will align the magnetic dipole in the direction of the applied magnetic field. This torque is responsible for **paramagnetism** in the materials.

We can consider each spinning electron to be a magnetic dipole but due to Pauli's exclusion principle the electrons of opposite spin are paired up so the net torque is zero. That's why paragmanetism is observed in materials with unpaired electrons.

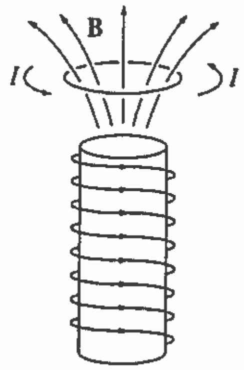
In a uniform magnetic field the net force on the magnetic dipole is zero:

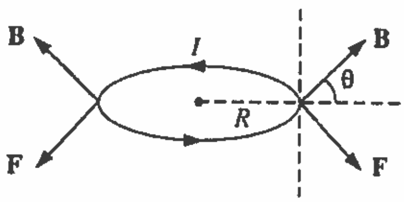
$$\vec{F} = i\oint d\vec{l} \times \vec{B} = i\left(\oint d\vec{l}\right) \times \vec{B} = 0$$

In a non-uniform field the net force on the magnetic dipole is not zero. Suppose a circular wire of radius R, carrying current I, is placed near the end of a short solenoid in the fringing region.

Here $\vec{B}$ has a radial component, so the magnitude of the force is:

$$F = 2\pi RIB\cos\theta$$





For an infinitesimal loop with dipole moment m, the force on the loop in a non-uniform magnetic field is therefore,

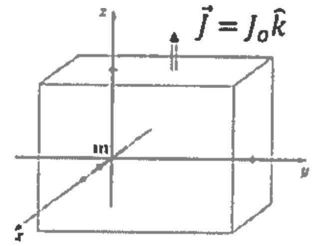
$$\vec{F} = \vec{\nabla}(\vec{m}.\vec{B})$$

Example 1:

A uniform current density $\vec{J} = J_o\hat{k}$ fills a slab straddling the yz-plane from $x = -a$ to $x = +a$. A magnetic dipole $\vec{m} = m_o\hat{\imath}$ is situated at the origin.

a) Find the force on the dipole.
b) Do the same for dipole pointing in the y-directions $\vec{m} = m_o\hat{\jmath}$
c) In the electrostatic case the expressions $\vec{F} = \vec{\nabla}(\vec{p}.\vec{E})$ and $\vec{F} = (\vec{p}.\vec{\nabla})\vec{E}$ are equivalent but this is not the case in magnetostatic, calculate $(\vec{m}.\vec{\nabla})\vec{B}$ for the configuration in (a) and (b).

Solution:

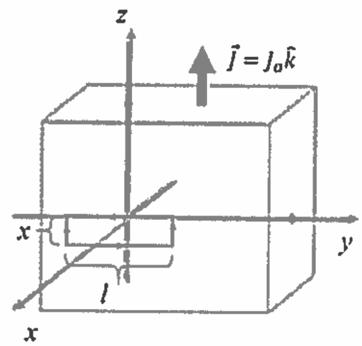


a)

$$\vec{F} = \vec{\nabla}(\vec{m}.\vec{B})$$

Lets first find the magnetic field due to the slab along x direction, using Ampere's law:

For the negative x values, $\vec{B}$ is pointed along –y-axis direction and for positive x values it is pointed along +y-axis direction. $\vec{B}$ will be zero at x=0



$$\oint \vec{B}.d\vec{l} = \mu_o i_{enc} = \mu_o J_o * xl$$

$$Bl = \mu_o J_o * xl$$

$$\vec{B} = \mu_o J_o x\hat{\jmath}$$

So $\vec{m}.\vec{B} = 0$ and hence $\vec{F} = 0$

b) For $\vec{m} = m_o\hat{\jmath}$

$$\vec{m}.\vec{B} = m_o\mu_o J_o x$$

$$\vec{F} = \vec{\nabla}(\vec{m}.\vec{B}) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)(m_o\mu_o J_o x) = m_o\mu_o J_o\hat{\imath}$$

c) $\vec{F} = \vec{\nabla}(\vec{p}.\vec{E})$ in electrostatics, let's start with the product rule in chapter 1.

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}.$$

$$\vec{\nabla}(\vec{p}.\vec{E}) = \vec{p}\times(\vec{\nabla}\times\vec{E}) + \vec{E}\times(\vec{\nabla}\times\vec{p}) + (\vec{p}.\vec{\nabla})\vec{E} + (\vec{E}.\vec{\nabla})\vec{p} = (\vec{p}.\vec{\nabla})\vec{E}$$

Because $\vec{\nabla}\times\vec{E} = 0$ and $\vec{p} = qd$ so derivative of $\vec{p}$ is zero that makes the second and fourth term equal to zero.

$$\vec{\nabla}(\vec{m}.\vec{B}) = \vec{m} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{m}) + (\vec{m}.\vec{\nabla})\vec{B} + (\vec{B}.\vec{\nabla})\vec{m}$$

$$(\vec{m}.\vec{\nabla})\vec{B} = \vec{\nabla}(\vec{m}.\vec{B}) - \vec{m} \times (\vec{\nabla} \times \vec{B}) - \vec{B} \times (\vec{\nabla} \times \vec{m}) - (\vec{B}.\vec{\nabla})\vec{m}$$

Since $\vec{\nabla} \times \vec{B} = \mu_o J \neq 0$

So

$(\vec{m}.\vec{\nabla})\vec{B} = \vec{\nabla}(\vec{m}.\vec{B}) - \vec{m} \times (\vec{\nabla} \times \vec{B})$ for a constant $\vec{m} = m_o\hat{\imath}$

For part (a):

$$(\vec{m}.\vec{\nabla})\vec{B} = 0 - m_o\hat{\imath} \times \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \mu_o J_o x & 0 \end{vmatrix} = -m_o\hat{\imath} \times [\mu_o J_o \hat{k}] = \mu_o m_o J_o \hat{\jmath}$$

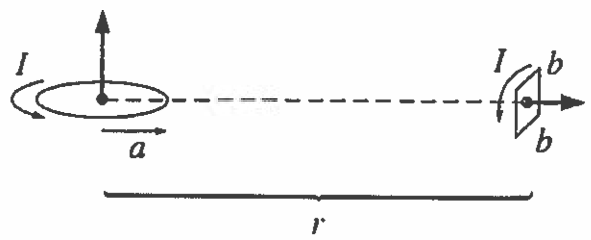
For part (b):

$$(\vec{m}.\vec{\nabla})\vec{B} = \left(m_o \frac{\partial}{\partial x}\right)(\mu_o J_o x \hat{\jmath}) = 0$$

Example 2:

a) Calculate the torque exerted on the square loop shown in the figure below due to the circular loop (assume r is much larger than a or b).

b) If the square loop is free to rotate, what will its equilibrium orientation be?



a) The dipole moment of the current loop is equal to

$$\vec{m} = \pi a^2 I\hat{k}$$

where we have defined the z axis to be the direction of the dipole. The magnetic field at the position of the square loop, at $r \gg a$, will be a dipole field with $\theta = 90°$:

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\boldsymbol{\phi}},$$

$$\mathbf{B}_{\text{dip}}(\mathbf{r}) = \nabla \times \mathbf{A} = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).$$

$$\vec{B} = \frac{\mu_o}{4\pi}\frac{m}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

For $\theta = 90°$

$$\vec{B} = \frac{\mu_o}{4\pi}\frac{m}{r^3}\hat{\theta}$$

$$\left.\begin{aligned}
\hat{\mathbf{r}} &= \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}, \\
\hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}, \\
\hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}},
\end{aligned}\right\}$$

For $\theta = 90°$

$$\vec{B} = -\frac{\mu_o}{4}\frac{a^2}{r^3}I\hat{k}$$

The dipole moment of the square loop is equal to

$$\vec{m}_{square} = b^2 I\hat{j}$$

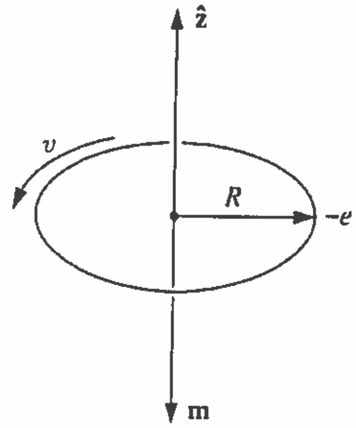
$$\vec{N} = \vec{m}_{square} \times \vec{B} = b^2 I\hat{j} \times \left(-\frac{\mu_o}{4}\frac{a^2}{r^3}I\hat{k}\right) = -\frac{\mu_o}{4}\frac{a^2 b^2}{r^3}I^2\hat{\imath}$$

b) In the equilibrium position, the torque on the current loop must be equal to zero. This therefore requires that dipole moment of the square loop should be in the same direction as the magnetic field that is along z-axis.

6.1.3. Effect of Magnetic Field on Atomic Orbitals

Electrons not only spin but also revolve around the nucleus at a very high frequency forming almost steady current. Consider a very classical picture of a Hydrogen atom consisting of an electron revolving in a circular orbit of radius R around a nucleus. Suppose that the velocity of the electron is equal to v. The current formed by the revoliving electron is:

$$I = \frac{e}{T} = \frac{ev}{2\pi R}$$



The dipole moment thus created is:

$$\vec{m} = I\vec{a} = -\frac{ev}{2\pi R} * \pi R^2 \hat{k} = -\frac{1}{2} evR\hat{k}$$

If the atom is placed in a magnetic field, it will be subject to a torque. It is very difficult to tilt the entire orbit, however there is a very significant effect on the orbital speed of the electron.

In the orbital motion, the centripetal force is provided by the electrical force alone:

$$\frac{1}{4\pi\varepsilon_o}\frac{e^2}{R^2} = m_e\frac{v^2}{R}$$

But in the presence of a magnetic field, the centripetal force will be sustained by both the electric and the magnetic field:

$$\frac{1}{4\pi\varepsilon_o}\frac{e^2}{R^2} + ev'B = m_e\frac{v'^2}{R}$$

Here we have assumed that the magnetic field is pointing along the positive z axis (in a direction opposite to the direction of the magnetic dipole moment). We have also assumed that the size of the orbit (r) does not change when the magnetic field is applied. Combining the last two equations we obtain

$$m_e\frac{v^2}{R} + ev'B = m_e\frac{v'^2}{R}$$

$$ev'B = \frac{m_e}{R}\left(v'^2 - v^2\right) = \frac{m_e}{R}(v' - v)(v' + v)$$

Assuming that the change in the velocity is small we can use the following approximations:

$$v' \cong v \text{ and } v' - v \cong \Delta v$$

$$evB = \frac{2m_e}{R} v\Delta v$$

$$\Delta v = \frac{eRB}{2m_e}$$

When the magnetic field is turned ON, the electron speeds up. A change in the orbital speed means a change in the dipole moment.

$$\Delta\vec{m} = -\frac{1}{2}e(\Delta v)R\hat{k} = -\frac{1}{2}e\left(\frac{eRB}{2m_e}\right)R\hat{k} = -\frac{e^2R^2}{4m_e}\vec{B}$$

Notice that the change in $\vec{m}$ is in the opposite direction of the applied magnetic field $\vec{B}$. An electron circling the other way would have a dipole moment pointing upward, but such an orbit would be slowed down by the field, so the change is still opposite to $\vec{B}$.

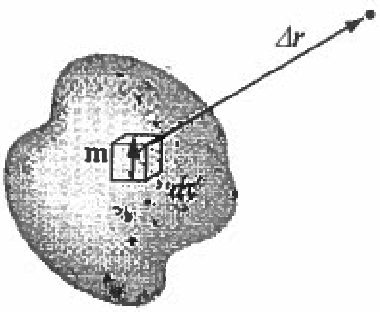
Ordinarily, the electron orbits are randomly oriented and the orbital dipole moments cancel out. But in the presence of a magnetic field, each atom picks up a little extra dipole moment, and these increments are all antiparallel to the applied field. This is the mechanism responsible for **diamagnetism.**

It is a universal phenomenon affecting all atoms. However, it is typically much weaker than paramagnetism, and is therefore observed mainly in atoms with even numbers of electrons, where paramagnetism is usually absent.

6.2. The Field of a Magnetized Object

6.2.1 Bound Currents

Consider a magnetized material with magnetization $\vec{M}$.
The associated vector potential $\vec{A}$ is given as:



$$\vec{A} = \frac{\mu_o}{4\pi}\frac{\vec{m}\times\Delta\hat{r}}{\Delta r^2}$$

Where $\Delta\vec{r} = \vec{r} - \vec{r}'$

In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\vec{M}d\tau'$, so the total vector potential is:

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int\frac{\vec{M}(\vec{r}')\times\Delta\hat{r}}{\Delta r^2}d\tau'$$

We can use the identity:

$$\vec{\nabla}'\left(\frac{1}{\Delta r}\right) = \frac{\Delta\hat{r}}{\Delta r^2}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int\left[\vec{M}(\vec{r}')\times\vec{\nabla}'\left(\frac{1}{\Delta r}\right)\right] d\tau'$$

Integrating by parts, we get

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\left\{\int\frac{1}{\Delta r}\left[\vec{\nabla}'\times\vec{M}(\vec{r}')\right] d\tau' - \int\vec{\nabla}'\times\left[\frac{\vec{M}(\vec{r}')}{\Delta r}\right] d\tau'\right\}$$

The second integral can be converted into surface integral as it was done in chapter-1.

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int\frac{1}{\Delta r}\left[\vec{\nabla}'\times\vec{M}(\vec{r}')\right] d\tau' + \frac{\mu_o}{4\pi}\oint\frac{1}{\Delta r}\left[\vec{M}(\vec{r}')\times d\vec{a}'\right]$$

The first term looks just like the potential of a volume current,

$$\vec{J}_b = \vec{\nabla}\times\vec{M}$$

While the second term looks like the potential of a surface current:

$$\vec{K}_b = \vec{M}\times\hat{n}$$

Where $\hat{n}$ is the normal unit vector. With these definitions:

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_V\frac{\vec{J}_b(\vec{r}')}{\Delta r}d\tau' + \frac{\mu_o}{4\pi}\oint\frac{\vec{K}_b(\vec{r}')}{\Delta r}da'$$

This means that potential and also the magnetic field of a magnetized object is the same as would be produced by a volume current $\vec{J}_b = \vec{\nabla}\times\vec{M}$ throughout the material plus the surface current $\vec{K}_b = \vec{M}\times\hat{n}$ on the boundary.

We first determine these bound currents and then find the field that they produce.

Example 3:

An infinitely long circular cylinder carries a uniform magnetization $\vec{M}$ parallel to its axis. Find the magnetic field (due to $\vec{M}$) inside and outside the cylinder.

Solution:

Consider a coordinate system S in which the z axis coincides with the axis of the cylinder. The magnetization of the material is:

$$\vec{M} = M\hat{k}$$

Since the material is uniformly magnetized, its bound volume current is equal to zero.

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$$

The bound surface current is equal to

$$\vec{K}_b = \vec{M} \times \hat{n} = \vec{M} \times \hat{r} = M\hat{\phi}$$

This current distribution is identical to the current distribution in an infinitely long solenoid.

The magnetic field outside an infinitely long solenoid is equal to zero, and therefore also the field outside the magnetized cylinder will be equal to zero.

The magnetic field inside an infinitely long solenoid can be calculated easily using Ampere's law and which is equal to:

$$\vec{B} = \mu_o K_b \hat{k} = \mu_o M\hat{k}$$

6.3. The Auxiliary Field *H*

The magnetic field in a system containing magnetized materials and free currents can be obtained by calculating the field produced by the total current $\vec{J}$ where,

$$\vec{J} = \vec{J}_{free} + \vec{J}_{bound}$$

This approach is very similar to the approach taken in electrostatics where the total electric field produced by a system containing dielectric materials is equal to the electric field produced by a charge distribution σ where

$$\sigma = \sigma_{free} + \sigma_{bound}$$

To calculate the magnetic field produced by a system containing magnetized materials we have to use the following form of Ampere's law:

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} = \mu_o\left(\vec{J}_{free} + \vec{J}_{bound}\right) = \mu_o\left(\vec{J}_{free} + \vec{\nabla} \times \vec{M}\right)$$

$$\vec{\nabla} \times \left(\frac{1}{\mu_o}\vec{B} - \vec{M}\right) = \vec{J}_{free}$$

The quantity in parenthesis is called the *H*-field

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M}$$

$\vec{H}$ plays a role in magnetostatics analogous to $\vec{D}$ in electrostatics. Ampere's law in terms of $\vec{H}$ can be written as:

$$\vec{\nabla} \times \vec{H} = \vec{J}_{free}$$

$$\oint \vec{H}.d\vec{l} = I_{free}$$

However, a knowledge of the free current density is not sufficient to determine $\vec{H}$. The Helmholtz theorem shows that besides knowing the curl of a vector function, we also need to know the divergence of that vector function before it is uniquely defined. Although the divergence of $\vec{B}$ is zero for any magnetic field (and therefore Ampere's law for $\vec{B}$ defines $\vec{B}$ uniquely) the divergence of $\vec{H}$ is not necessarily zero:

$$\vec{\nabla}.\vec{H} = \vec{\nabla}.\left(\frac{1}{\mu_o}\vec{B} - \vec{M}\right) = \frac{1}{\mu_o}\vec{\nabla}.\vec{B} - \vec{\nabla}.\vec{M} = -\vec{\nabla}.\vec{M}$$

Therefore, only for those systems where $\vec{\nabla}.\vec{M} = 0$ we can use Ampere's law for $\vec{H}$ directly to calculate $\vec{H}$. The divergence of $\vec{H}$ will be zero only for systems with cylindrical, plane, solenoidal, or toroidal symmetry.

The $\vec{H}$ field is a quantity that is used in the laboratory more often that the $\vec{B}$ field. This is a result of the dependence of $\vec{H}$ on only the free currents (which are easy to control).

The $\vec{B}$ field depends both on the free and on the bound currents, and thus requires a detailed knowledge of the magnetic properties of the materials used.

In electrostatics, the electric field can be obtained immediately from the potential difference (which is easy to control). The electric displacement $\vec{D}$ however depends only on the free charge distribution, but in most cases a direct measurement of the free charge distribution is very difficult to carry out.

Therefore, in electrostatics the electric field is in most cases a more useful parameter then the electric displacement $\vec{D}$.

Example 4:

An infinitely long cylinder, of radius R, carries a "frozen-in" magnetization, parallel to the axis, $\vec{M} = kr\hat{k}$, where k is a constant and r is the distance from the axis (there is no free current anywhere). Find the magnetic field inside and outside the cylinder by two different methods:

a) Locate all the bound currents, and calculate the field they produce.

b) Use Ampere's law to find $\vec{H}$, and then get $\vec{B}$.

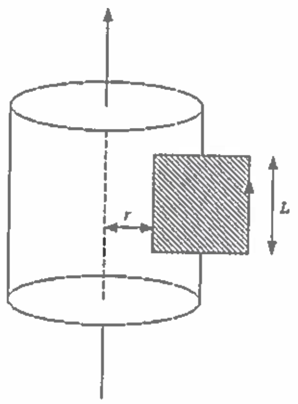
Solution:

a) The bound volume current is equal to

$$\vec{J}_b = \vec{\nabla} \times \vec{M} = \frac{1}{r}\frac{\partial M}{\partial z}\hat{r} - \frac{\partial M}{\partial r}\hat{\phi} = -k\hat{\phi}$$

The bound surface current is equal to

$$\bar{K}_b = \vec{M} \times \hat{n}\big|_{r=R} = kR\hat{k} \times \hat{r} = kR\hat{\phi}$$



The bound currents produce a solenoidal field. The field outside the cylinder will be equal to zero and the field inside the cylinder will be directed along the z axis. Its magnitude can be obtained using Ampere's law.

Consider the Amperian loop shown in the figure below. The line integral of $\vec{B}$ along the Amperian loop is equal to

$$\oint \vec{B}.d\vec{l} = \mu_o i_{enc}$$

$$\oint \vec{B}.d\vec{l} = -BL$$

The current intercepted by the Amperian loop is:

$$i_{enc} = -K_b L + \int_r^R J_b L\, dr = -KLR + \int_r^R KL\, dr = -KLR + KL(R - r) = -KLr$$

Ampere's law can now be used to calculate the magnetic field:

$$\vec{B} = \frac{\mu_o i_{enc}}{-L}\hat{k} = \mu_o k r \hat{k}$$

b) The divergence of $\vec{M}$ is equal to zero. Therefore, Ampere's law uniquely defines $\vec{H}$. The $\vec{H}$ field is pointing in the z-direction.

Using Ampere's law, in terms of the $\vec{H}$ field, we immediately conclude that for the Amperian loop shown in the figure above.

$$\oint \vec{H}.d\vec{l} = HL = i_{free} = 0$$

since there is no free current This can only be true if $\vec{H} = 0$. This implies that

$$\vec{H} = \frac{1}{\mu_o}\vec{B} - \vec{M} = 0$$
$$\vec{B} = \mu_o \vec{M}$$

In the region outside the cylinder the magnetization is equal to zero and therefore the magnetic field

$$\vec{B}_{outside} = 0$$

In the region inside the cylinder the magnetization is equal to

$$\vec{M} = kr\hat{k}$$

and therefore the magnetic field is equal to

$$\vec{B} = \mu_o \vec{M} = \mu_o k r \hat{k}$$

which is identical to the result obtained in part a).

Example 5:

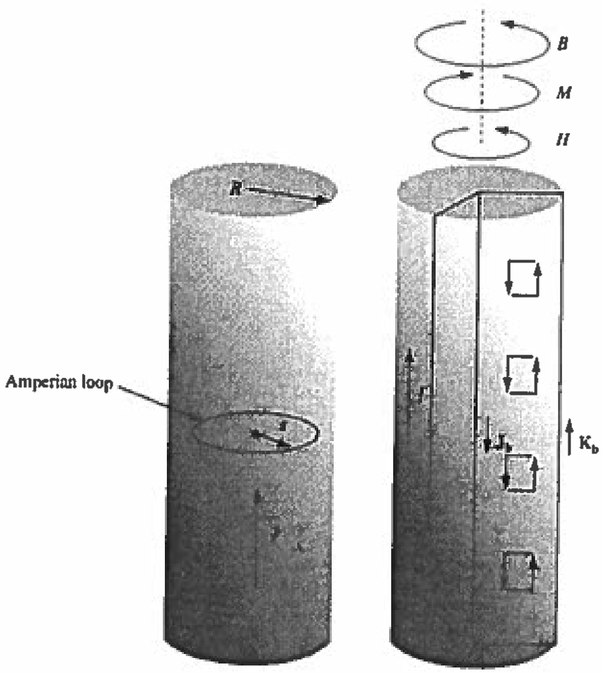
A long copper wire of radius R carries a uniformly distributed (free) current i. Find H inside and outside the rod.

Solution:

Copper is weakly diamagnetic material so the bound current will run in the opposite direction to the free current within the rod and along the direction of free current on the surface.

Since all the currents are along the axis of the rod so B, M and H are all circumferential.

Applying Ampere's law for the auxiliary field H, we get:



$$H(2\pi s) = I_{free} = I\frac{\pi s^2}{\pi R^2}$$

$$\vec{H} = \frac{Is}{2\pi R^2}\hat{\phi} \quad for\ s \leq R$$

Within the wire and outside the wire, using Ampere's law, we get:

$$\vec{H} = \frac{I}{2\pi s}\hat{\phi} \quad for\ s \geq R$$

In the outside region $\vec{M} = 0$, hence

$$\vec{B} = \mu_o\vec{H} = \frac{\mu_o I}{2\pi s}\hat{\phi} \quad for\ s \geq R$$

Inside the region, we cannot determine the magnetic field because we don't know what is $\vec{M}$ in the wire.

Example 6:

Suppose the field inside a large piece of material is $\vec{B}_o$, and the corresponding $\vec{H}_o$ field is equal to:

$$\vec{H}_o = \frac{1}{\mu_o}\vec{B}_o - \vec{M}$$

A small spherical cavity is hollowed out of the material. Find the $\vec{B}$ field at the center of the cavity in terms of $\vec{B}_o$ and $\vec{M}$. Also find the $\vec{H}$ field at the center of the cavity in terms of $\vec{H}_o$ and $\vec{M}$. Assume the cavity is small enough so that $\vec{M}$, $\vec{B}_o$, and $\vec{H}_o$ are essentially constant.

Solution:

The field in the spherical cavity is the superposition of the field $\vec{B}_o$ and the field produced by a sphere with magnetization $-\vec{M}$. The bound volume current in the sphere is equal to zero (uniform magnetization). The bound surface current is equal to

$$\vec{K}_b = \left(-\vec{M}\right) \times \hat{n} = -M\sin\theta\,\hat{\phi}$$

Here we have assumed that the magnetization of the sphere is directed along the *z*-axis. Now consider a uniformly charged sphere, rotating with an angular velocity ω around the *z*-axis. The system carries a surface current equal to:

$$\vec{K} = \sigma\vec{v} = \sigma R\omega\sin\theta\;\hat{\phi}$$

Comparing these two equations for the surface current, we conclude that

$$M = -\sigma R\omega$$

In Chapter 5 the magnetic field produced by a uniformly charged, rotating sphere was calculated. The magnetic field inside the sphere was found to be uniform and equal to

$$\vec{B} = \frac{2}{3}\mu_o\sigma\omega R\hat{k} = -\frac{2}{3}\mu_o\vec{M}$$

The magnetic field inside the spherical cavity is therefore equal to

$$\vec{B}_{cavity} = \vec{B}_o + \vec{B}_{sphere} = \vec{B}_o - \frac{2}{3}\mu_o\vec{M}$$

The corresponding $\vec{H}$ field is equal to:

$$\vec{H}_{cavity} = \frac{1}{\mu_o}\vec{B}_{cavity} - \vec{M}_{cavity} = \frac{1}{\mu_o}\vec{B}_o - \frac{2}{3}\vec{M} = \vec{H}_o + \frac{1}{3}\vec{M}$$

Here we have used the fact that $\vec{M}_{cavity} = 0$ since no materials are present there.

6.4. Linear Media

In paramagnetic and diamagnetic materials, the magnetization is maintained by the external magnetic field. The magnetization disappears when the field is removed. Most paramagnetic and diamagnetic materials are linear; that is their magnetization is proportional to the $\vec{H}$ field:

$$\vec{M} = \chi_m \vec{H}$$

The constant of proportionality χ_m is called the magnetic susceptibility of the material. In vacuum, the magnetic susceptibility is zero.

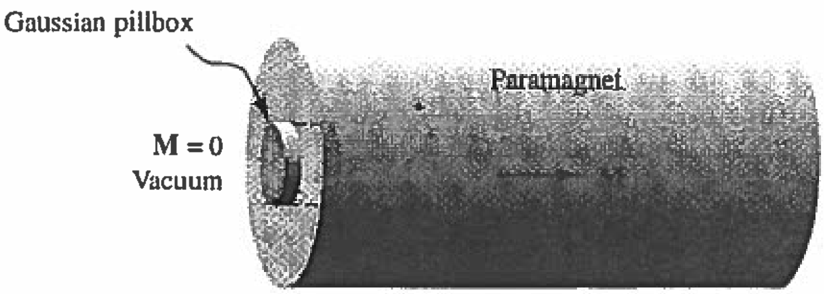
In a linear medium, there is linear relation between the magnetic field and the $\vec{H}$ field:

$$\vec{B} = \mu_o(\vec{H} + \vec{M}) = \mu_o(1 + \chi_m)\vec{H} = \mu\vec{H}$$

Where $\mu = \mu_o(1 + \chi_m)$ is called the permeability of the material. The permeability of free space is equal to μ_o.

The linear relation between $\vec{H}$ and $\vec{B}$ does not automatically imply that the divergence of $\vec{H}$ is zero. The divergence of $\vec{H}$ will only be equal to zero inside a linear material, but will be non-zero at the interface between two materials of different permeability. Consider for example the interface between a linear material and vacuum (see Figure below). The surface integral of $\vec{M}$ across the surface of the Gaussian pillbox shown in the figure is definitely not equal to zero. According to the divergence theorem the surface integral of $\vec{M}$ is equal to the volume integral of $\vec{\nabla}.\vec{M}$:

$$\oint \vec{M}.d\vec{a} = \int (\vec{\nabla}.\vec{M})d\tau$$

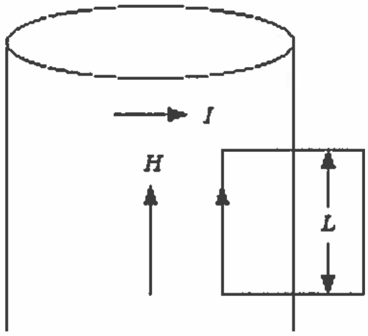


Therefore, if the surface integral of $\vec{M}$ is not equal to zero, the divergence of $\vec{M}$ can not be zero everywhere.

Example 7:

An infinite solenoid (N turns per unit length, current I) is filled with linear material of susceptibility χ_m. Find the magnetic field inside the solenoid.

Solution:



Because of the symmetry of the problem, the divergence of $\vec{H}$ will be equal to zero, everywhere. Therefore, the $\vec{H}$ field can be obtained directly from Ampere's law. Consider the Amperian loop shown in the figure above. The line integral of $\vec{H}$ around the loop is:

$$\oint \vec{H}.d\vec{l} = HL$$

Where the line integral is evaluated in the direction shown in the figure, and it is assumed that the $\vec{H}$ field is directed along the z-axis. The free current intercepted by the Amperian loop is equal to

$$I_{free} = NIL$$

Ampere's law for the $\vec{H}$ field immediately shows that

$$\vec{H} = NI\hat{k}$$

The magnetic field inside the solenoid is equal to

$$\vec{B} = \mu_o(1 + \chi_m)\vec{H} = \mu_o(1 + \chi_m)NI\hat{k}$$

The magnetization of the material is equal to

$$\vec{M} = \chi_m\vec{H} = \chi_m NI\hat{k}$$

and is uniform. Therefore, there will be no bound volume currents in the material. The bound surface current is equal to

$$\vec{K}_b = \vec{M} \times \hat{n} = \chi_m NI\hat{k} \times \hat{n} = \chi_m NI\hat{\phi}$$

This last equation shows that the bound surface current flows in the same direction (paramagnetic materials) or in an opposite direction (diamagnetic materials) as the free current.

6.5. Nonlinear Media

The best known nonlinear media are the ferromagnetic materials. Ferromagnetic materials do not require external fields to sustain their magnetization (therefore, the magnetization definitely depends in a nonlinear way on the field). The magnetization in ferromagnetic materials involves the alignment of the dipole moments associated with the spin of unpaired electrons.

The difference between ferromagnetic materials and paramagnetic materials is that in ferromagnetic materials the interaction between nearby dipoles makes them want to point in the same direction, even when the magnetic field is removed. However, the alignment occurs in relative small patches, called **domains**. When a ferromagnetic material is not located in a magnetic field, the dipole moments of the various domains are not aligned, and the material as a whole is not magnetized. When the ferromagnetic material is put into a magnetic field, the boundaries of the domain parallel to the field will increase at the expense of neighboring boundaries. If the field is strong enough, one domain takes over entirely, and the ferromagnetic material is said to be saturated (all unpaired electrons are aligned and therefore the magnetization reaches a maximum value).

The magnetic susceptibility of ferromagnetic materials is around $\mathbf{10^3}$ (roughly eighth orders of magnitude larger than the susceptibility of paramagnetic materials).

When the magnetic field is removed some magnetization remains (and we have created a permanent magnet). For any ferromagnetic material, the magnetization depends not only on the applied magnetic field but also on the magnetization history.

The alignment of dipoles in a ferromagnet can be destroyed by random thermal motion. The destruction of the alignment occurs at a precise temperature (called the **Curie point**). When a ferromagnetic material is heated above its Curie temperature it becomes paramagnetic.