Chapter 5. Magnetostatics

5.1. The Magnetic Field

Consider two parallel straight wires in which current is flowing. The wires are neutral and therefore there is no net electric force between the wires.

Nevertheless, if the current in both wires is flowing in the same direction, the wires are found to attract each other. If the current in one of the wires is reversed, the wires are found to repel each other.

The force responsible for the attraction and repulsion is called the **magnetic force**. The magnetic force acting on a moving charge q is defined in terms of the **magnetic field**:

$$\vec{F} = q(\vec{v} \times \vec{B})$$

In a region where there is an electric field and a magnetic field the total force on the moving force is:

$$\vec{F}_{total} = \vec{F}_{electric} + \vec{F}_{magnetic} = q\vec{E} + q(\vec{v} \times \vec{B})$$

This equation is called the **Lorentz force law** and provides us with the total electromagnetic force acting on q.

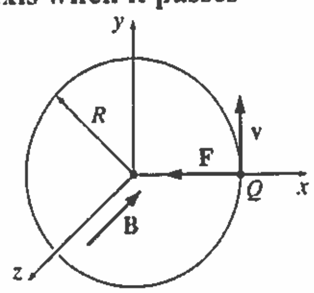
An important difference between the electric field and the magnetic field is that the electric field does work on a charged particle (it produces acceleration or deceleration) while the magnetic field does not do any work on the moving charge. This is a direct consequence of the Lorentz force law:

$$dW_{magnetic} = \vec{F}_{magnetic} \cdot d\vec{l} = q[\vec{v} \times \vec{B}].\vec{v}dt = 0$$

We conclude that the magnetic force can alter the direction in which a particle moves, but cannot change its velocity.

Example 1: Cyclotron Motion

A particle of mass m and charge q is moving with velocity along y-axis when it passes through a uniform magnetic field B pointing in the negative z-direction. Find the radius of the circle the particle will be moving and find its frequency as well.



$$qvB = \frac{mv^2}{R}$$

$$R = \frac{mv}{qB}$$

Time period (T):

$$T = \frac{2\pi R}{v} = \frac{2\pi m}{qB}$$

Cyclotron frequency (ω):

$$\omega = \frac{2\pi}{T} = \frac{qB}{m}$$

$$\ddot{y} = \omega \dot{z} \quad , \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

$$\dot{y} = \omega z + c_1 .$$

$$\ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right) = \omega \frac{E}{B} - \omega (\omega z + c_1)$$

$$\ddot{z} = \omega \frac{E}{B} - \omega^2 z - c_1 \omega$$

$$\ddot{z} = -\omega^2 z + \left(\frac{E}{B} - c_1\right)\omega$$

$$z = c_2 e^{i\omega t} + \left(\frac{E}{B} - c_1\right)\omega =$$

$$\frac{dy}{dt} = \omega \left[c_2 e^{i\omega t} \right] + \left(\frac{E}{B} - c_1\right)\omega^2 + c_1$$

$$y = \cancel{\omega} c_2 \frac{e^{i\omega t}}{i\cancel{\omega}} + \left(\frac{E}{B} - c_1\right)\omega^2 t + c_1 t + c_3$$

$$y = -i c_2 e^{i\omega t} + \left(\frac{E}{B}\omega^2 - c_1 \omega^2 + c_1\right) t + c_3$$

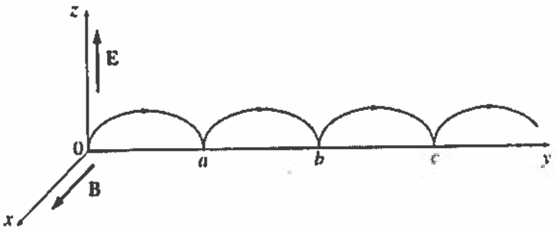
$$\dot{y} = \omega ($$

Example 2: Cycloid Motion

A positively charged particle at rest is released from the origin where a uniform electric field is pointing in the z-direction and a uniform magnetic field is pointing in the x-direction. Find the trajectory of the particle.

Solution:

Initially the particle is at rest so no magnetic force and electrical force will accelerate it in the z-direction. As the particle will pick-up the speed magnetic force will start acting on it and as the speed gets higher and higher the magnetic force will increase and force it to bend in the negative z-direction and this will reduce the speed of the particle because acceleration due to electric force is in the opposite direction and on the y-axis its speed will be zero and hence no more magnetic force. Then it will pick-up speed again in the z-direction and cycle will continue, and the particle will follow the trajectory shown in the figure below:



Since there would be no force in the x-direction, so position of the particle at any time can be expressed as follows:

$$\vec{r}(0, y(t), z(t))$$

And the velocity of the particle will be:

$$\vec{v}(0, \dot{y}(t), \dot{z}(t))$$

Thus:

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = (B\dot{z})\hat{\jmath} - (B\dot{y})\hat{k}$$

Using Newtons' second law:

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}) = m\vec{a} = m(\ddot{y}\hat{\jmath} + \ddot{z}\hat{k})$$

$$qE\hat{k} + q(B\dot{z}\hat{\jmath} - B\dot{y}\hat{k}) = m\vec{a} = m(\ddot{y}\hat{\jmath} + \ddot{z}\hat{k})$$

$$qB\dot{z} = m\ddot{y} \quad \text{and} \quad qE - qB\dot{y} = m\ddot{z}$$

We can use:

$$\omega = \frac{qB}{m}$$

Which is a cyclotron frequency we calculated in example1, in the absence of an applied electric field. So,

$$\ddot{y} = \omega\dot{z}$$

$$\ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

The general solution for these equations is:

$$y(t) = C_1\cos(\omega t) + C_2\sin(\omega t) + \left(\frac{E}{B}\right)t + C_3$$

$$\ddot{y} = \omega \dot{z} = \omega(-c_2\omega \sin(\omega t) - c_1\omega \cos(\omega t))$$

$$\ddot{y} = -c_2\omega^2 \sin(\omega t) - c_1\omega^2 \cos(\omega t)$$

$$\frac{dy}{dt} = +c_2\omega \cos(\omega t) - c_1\omega \sin(\omega t) + c_3$$

$$y = +c_2 \sin(\omega t) + c_1 \cos(\omega t) + c_3 t$$

$$\left\{\begin{array}{l} y = c_1 \cos(\omega t) + c_2 \sin(\omega t) + c_3 t + c_4 \\ z = c_2 \cos(\omega t) - c_1 \sin(\omega t) + c_5 \end{array}\right)$$

at $t=0$

$y = 0 = c_1 + 0 + 0 + c_4 \Rightarrow c_4 = 0$

$z = 0 = c_2 - 0 + c_5 \Rightarrow c_2 = -c_5$ $\qquad c_5 = \frac{c_3}{\omega}$

at $t=0$ $\qquad \Rightarrow c_2 = -\frac{c_3}{\omega}$

$\dot{y} = 0 = 0 + \omega c_2 + c_3$

$\dot{z} = 0 = 0 - \omega c_1 \Rightarrow c_1 = 0$

$$y = -\frac{c_3}{\omega} \sin \omega t + c_3 t = -\frac{E}{B\omega}[\omega t - \sin \omega t]$$

$$z = \frac{c_3}{\omega} \cos(\omega t) + \frac{c_3}{\omega} = \frac{E}{B\omega}[1 - \cos(\omega t)]$$

$$z(t) = C_2 \cos(\omega t) - C_1 \sin(\omega t) + C_4$$

Applying the initial conditions that at $t = 0$, $y(0) = z(0) = 0$ and $\dot{y}(0) = \dot{z}(0) = 0$

We get:

$$y(t) = \frac{E}{\omega B}[\omega t - \sin(\omega t)]$$

$$z(t) = \frac{E}{\omega B}[1 - \cos(\omega t)]$$

If we let:

$$R = \frac{E}{\omega B}$$

$$\sin(\omega t) = \omega t - \frac{y(t)}{R}$$

$$\cos(\omega t) = 1 - \frac{z(t)}{R}$$

Squaring both equations and adding them, we get:

$$(y(t) - R\omega t)^2 + (z(t) - R)^2 = R^2$$

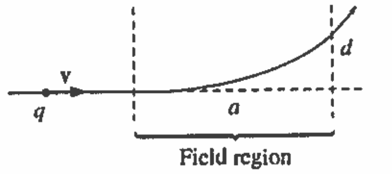
Which is an equation of a circle whose center $(0, R\omega t, R)$ travels along y-axis with a constant speed of:

$$v = \omega R = \omega \frac{E}{\omega B} = \frac{E}{B}$$

The particle moves down the y-axis as a point on the rim of a wheel, rolling down y-axis with a constant speed $v = \frac{E}{B}$. The curve generated is called **cycloid**.

Example3:

A particle of charge q enters the region of uniform magnetic field $\vec{B}$ (pointing into the page). The field deflects the particle a distance d above the original line of flight, as shown in the figure. Is the charge positive or negative? In terms of a, d, B, and q, find the momentum of the particle.



Using Lorentz force law and right hand rule the charge of the particle is positive so it can deflect upwards.

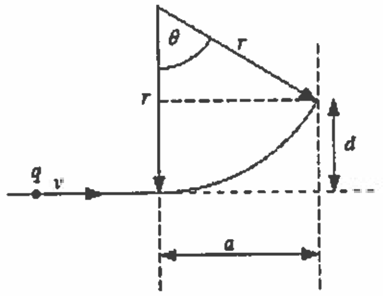
The magnitude of the force acting on the moving charge is:

$F = qvB = \frac{mv^2}{r}$

$$p = mv = qBr$$

where p is the momentum of the particle.

Figure below shows the following relation between r, d and a:



$$(r - d)^2 + a^2 = r^2$$

This equation can be used to express r in terms of d and a:

$$r = \frac{d^2 + a^2}{2d}$$

The momentum of the charge q is therefore equal to

$$p = qBr = qB\left(\frac{d^2 + a^2}{2d}\right)$$

5.1.3 Magnetic Force on Currents

The electric current in a wire is due to the motion of the electrons in the wire. If the current flows in the presence of an external magnetic field then each electron will experience a magnetic force. Consider a tiny segment of the wire of length *dl* and assume the linear electron density is $-\lambda$ C/m and each electron is moving with a velocity *v*. The magnetic force exerted by the magnetic field on a single electron will be:

$$d\vec{F}_e = -e(\vec{v} \times \vec{B})$$

A segment of the wire of length *dl* contains λ**dl/e* electrons. Therefore the magnetic force acting on this segment will be:

$$d\vec{F}_{magnetic} = \frac{\lambda dl}{e} d\vec{F}_e = I(d\vec{l} \times \vec{B})$$

Where, $I = \frac{dq}{dt} = \frac{dq}{dl}\frac{dl}{dt} = \lambda v$

The direction of $d\vec{l}$ is the direction of the current and opposite to the direction of velocity of electrons. The total force on the wire would be:

$$\vec{F}_{magnetic} = \int_{wire} I(d\vec{l} \times \vec{B}) = I \int_{wire} (d\vec{l} \times \vec{B})$$

Here we assume that the current is constant throughout the wire.

If the current is flowing over a surface, it is usually described by a **surface current density** $\vec{K}$, which is current per unit length-perpendicular-to-flow. The force on a surface current is:

$$\vec{F}_{magnetic} = \int_{surface} (\vec{K} \times \vec{B}) da$$

If the current flows through a volume, is it is usually described in terms of a **volume current density** $\vec{J}$. The magnetic force on a volume current is equal to

$$\vec{F}_{magnetic} = \int_{Volume} (\vec{J} \times \vec{B}) d\tau$$

Since $\vec{J} = \frac{dI}{da_\perp}$ so $I = \int_{surface} dI = \int_S J\, da_\perp = \int_S \vec{J}.d\vec{a} = \int_V \vec{\nabla}.\vec{J}\, d\tau$

So the total current (or total charge per unit volume leaving the volume) is:

$$I = \int_V \vec{\nabla}.\vec{J}\, d\tau = -\frac{dq}{dt} = -\frac{d}{dt}\int_V \rho\, d\tau = -\int_V \frac{\partial\rho}{\partial t} d\tau$$

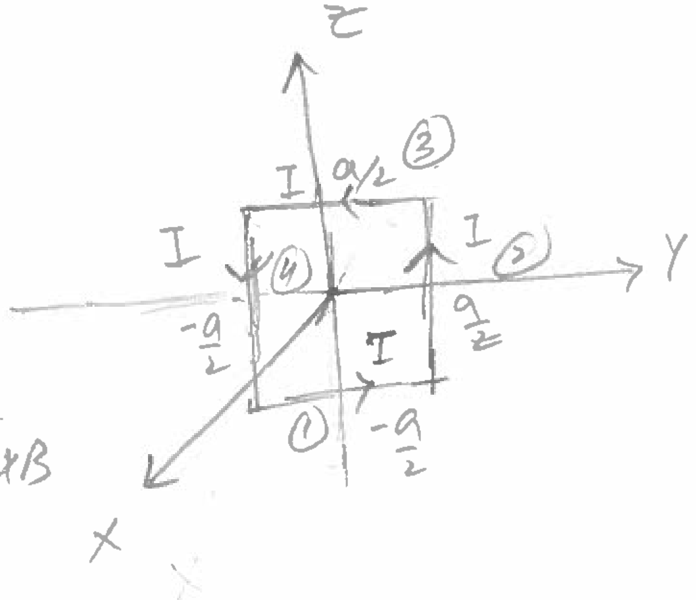
The negative sign indicates that the charge is leaving the surface and left over charge in the volume is decreasing. The above equation also implies that:

$$\vec{\nabla}.\vec{J} = -\frac{d\rho}{dt}$$

This equation is known as the **continuity equation**.

Example 4:

Suppose the magnetic field in some region has the form $\vec{B} = kz\hat{\imath}$, where k is a constant. Find the force on a square loop of side a, lying in the yz-plane and centered at the origin, if it carries a current, flowing counter clockwise when you look down the x-axis.



$$\bar{F} = i \int d\bar{l} \times \bar{B}$$

$$= I \left[\int_1 d\bar{l} \times \bar{B} + \int_2 d\bar{l} \times \bar{B} + \int_3 d\bar{l} \times \bar{B} + \int_4 d\bar{l} \times \bar{B} \right]$$

$$= I \left[\int_{-\frac{a}{2}}^{\frac{a}{2}} dy\, kz\, \hat{j} \times \hat{i} \Big|_{z=-\frac{a}{2}} + \int_{-\frac{a}{2}}^{a/2} dz\, kz\, \hat{k} \times \hat{i} + \int_{a/2}^{-a/2} dy\, kz\, \hat{j} \times \hat{i} \Big|_{z=a/2} + \int_{\frac{a}{2}}^{-a/2} dz\, kz\, \hat{k} \times \hat{i} \right]$$

$$= I \left[k\left(-\frac{a}{2}\right)(-\hat{k})\left(\frac{a}{2} - \left(-\frac{a}{2}\right)\right) + k(\hat{j})\underbrace{\cancel{\left(\left(\frac{a}{2}\right)^2 - \left(-\frac{a}{2}\right)^2\right)}}_{=0} + k\left(\frac{a}{2}\right)(-\hat{k})\left(-\frac{a}{2} - \frac{a}{2}\right) + k(\hat{j})\underbrace{\cancel{\left(\left(-\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right)}}_{=0} \right]$$

$$= I \left[k\frac{a^2}{2}\hat{k} + 0 + \frac{ka^2}{2}\hat{k} + 0 \right] = \boxed{Ika^2\hat{k} = \bar{F}}$$

$$B(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{S}{\cos^2\theta} d\theta \times \frac{\cos^2\theta}{S^2} = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\cos\theta}{S} d\theta$$

$$\boxed{B(\vec{r}) = \frac{\mu_0 I}{4\pi S}(\sin\theta_2 - \sin\theta_1)}$$

where S is the perpendicular distance from the wire to the field point.

for an infinite wire $\theta_1 = \frac{-\pi}{2}$ and $\theta_2 = \frac{\pi}{2}$

$$B(r) = \frac{\mu_0 I}{4\pi S}\left(\sin\frac{\pi}{2} + \sin\frac{\pi}{2}\right) = \boxed{\frac{\mu_0 I}{2\pi S}}$$

for a semi-infinite wire $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$

$$B(r) = \frac{\mu_0 I}{4\pi S}\left(\sin\frac{\pi}{2} - \sin(0)\right) = \boxed{\frac{\mu_0 I}{4\pi S}}$$

5.2. The Biot-Savart Law

A steady current is a flow of charge that has been going on forever, and will be going on forever. These currents produce magnetic fields that are constant in time.

The magnetic field produced by a steady line current is given by the **Biot-Savart Law**:

$$\vec{B}(P) = \frac{\mu_o I}{4\pi}\int \frac{d\vec{l} \times \Delta\hat{r}}{\Delta r^2}$$

where $d\vec{l}$ is an element of the wire, $\hat{r}$ is the vector connecting the element of the wire and the field point P, and μ_o is the permeability constant $[\mu_o = 4\pi \times 10^{-7}\ N/A^2]$

The SI unit of the magnetic field is **Tesla** (T). For surface and volume currents the Biot-Savart law can be written as:

$$\vec{B}(P) = \frac{\mu_o I}{4\pi}\int_S \frac{\vec{K} \times \Delta\hat{r}}{\Delta r^2}\, da$$

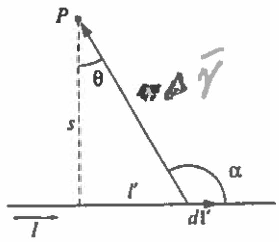
$$\vec{B}(P) = \frac{\mu_o I}{4\pi}\int_V \frac{\vec{J} \times \Delta\hat{r}}{\Delta r^2}\, d\tau$$

direction of $\bar{B}$ is given by the right hand rule.

Example 5:

Find the magnetic field at a distance s from a long straight wire carrying a steady current I as shown in the figure below:

Solution:



$$\bar{B}(\bar{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\bar{l}' \times \Delta\bar{r}}{\Delta r^2}$$

$$|d\bar{l}' \times \hat{\Delta r}| = dl' \sin\alpha = dl' \sin(\theta + \tfrac{\pi}{2}) = dl' \cos\theta$$

$$l' = \Delta r \sin\theta \Leftrightarrow \Delta r = \frac{l'}{\sin\theta}$$

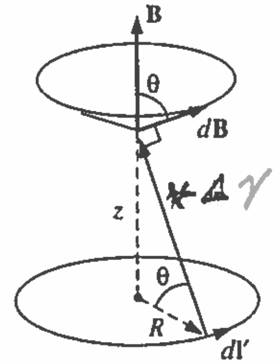
$$\frac{l'}{s} = \tan\theta \Rightarrow l' = s\tan\theta \Rightarrow \Delta r = \frac{s\tan\theta}{\sin\theta} = \frac{s}{\cos\theta}$$

$$\frac{1}{\Delta r^2} = \frac{\cos^2\theta}{s^2}$$

$$dl' = \frac{s}{\cos^2\theta}\, d\theta \quad \text{so} \quad |d\bar{l}' \times \hat{\Delta r}| = dl'\cos\theta = \frac{s\, d\theta}{\cos\theta}$$

Example 6: Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I.

Solution:



$$B(z) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \Delta \hat{r}}{\Delta r^2}$$

At distance z from the loop horizontal components of B cancels out and vertical components survive, so

$$B(z) = \frac{\mu_0 I}{4\pi} \int \frac{dl'}{\Delta r^2} \cos\theta$$

$$\Delta r^2 = R^2 + z^2 \quad \text{and} \quad \int dl' = 2\pi R$$

$R = \Delta r \cos\theta$

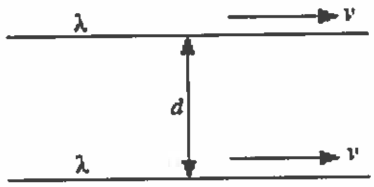
$\cos\theta = \frac{R}{\Delta r} = \frac{R}{\sqrt{R^2+z^2}}$

$$B(z) = \frac{\mu_0 I}{\cancel{4\pi}_{\,2}} \frac{\cancel{2\pi} R}{(R^2+z^2)} \times \frac{R}{\sqrt{R^2+z^2}}$$

$$\boxed{B(z) = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}}$$

at $z=0$ $\quad B(0) = \frac{\mu_0 I R^2}{2R^3} = \boxed{\frac{\mu_0 I}{2R}}$

Example 7:
Suppose you have two infinite straight-line charges λ, a distance d apart, moving along at a constant speed v. How fast should the speed of charges be in order for the magnetic attraction to balance the electrical repulsion?



When a line charge moves it looks like a current of magnitude $I = \lambda v$. The two parallel currents attract each other, and the attractive force per unit length is:

$$f_B = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d} = \frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d}$$

and is attractive. The electric field generated by one of the wires is given by Gauss' law:

$$E(r) = \frac{1}{2\pi\varepsilon_o}\frac{\lambda}{r}$$

The electric force per unit length acting on the other wire is equal to

$$f_E = \lambda E(d) = \frac{1}{2\pi\varepsilon_o}\frac{\lambda^2}{d}$$

and is repulsive (like charges).

The electric and magnetic forces are balanced when:

$$f_B = f_E$$

$$\frac{\mu_0}{2\pi}\frac{\lambda^2 v^2}{d} = \frac{1}{2\pi\varepsilon_o}\frac{\lambda^2}{d}$$

$$\mu_0 v^2 = \frac{1}{\varepsilon_0} \rightarrow v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 m/s$$

This requires that the speed v is equal to the speed of light, and this can therefore never be achieved. Therefore, at all velocities the electric force will dominate.

5.3. The Divergence and Curl of *B*.

Using the Biot-Savart law for a current density $\vec{J}$ we can calculate the divergence and curl of $\vec{B}$:

Proof:

In cylindrical coordinates:

$$\vec{\nabla}.\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial}{\partial \phi}(v_\phi) + \frac{\partial v_z}{\partial z}$$

We can use the example of magnetic field due to an infinite long straight wire:

$$\vec{B} = \frac{\mu_o i}{2\pi s}\hat{\phi}$$

$$\vec{\nabla}.\vec{B} = \frac{1}{s}\frac{\partial}{\partial \phi}\left(\frac{\mu_o i}{2\pi s}\right) = 0$$

$$\boxed{\vec{\nabla}.\vec{B} = 0}$$

Similarly:

$$\int (\vec{\nabla}\times\vec{B}).d\vec{a} = \oint \vec{B}.d\vec{l} = \oint \frac{\mu_o i}{2\pi s} s d\phi = \mu_o i = \mu_o \int \vec{J}.d\vec{a}$$

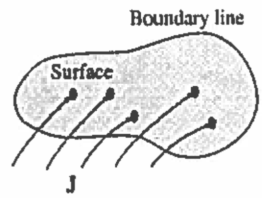
$$\boxed{\vec{\nabla}\times\vec{B} = \mu_o\vec{J}}$$

This last equation is called **Ampere's law in differential form**. This equation can be rewritten, using Stokes' law, as

$$\oint \vec{B}.d\vec{l} = \mu_o i_{enclosed}$$

This equation is called **Ampere's law in integral form.** The direction of evaluation of the line integral and the direction of the surface element vector $d\vec{a}$ must be consistent with the right-hand rule.

We can use the positive direction of current by using the right hand rule, we curl our fingers around the loop for integration then extended thumb direction will be the direction of positive current.



Ampere's law is always true, but is only a useful tool to evaluate the magnetic field if the symmetry of the system enables you to pull $\vec{B}$ outside the line integral. The configurations that can be handled by Ampere's law are:

1. Infinite straight lines
2. Infinite planes
3. Infinite solenoids
4. Toroids

Example 8:

Use the Ampere's law to find the magnetic field due to current in a straight infinite wire.

Solution:

We can make an Amperian loop in the form of circle around the straight wire and direction of B will be circumferential and constant over the loop. So

$$\oint \vec{B}.d\vec{l} = B\int_0^{2\pi} s d\phi = Bs2\pi = \mu_o i_{enclosed}$$

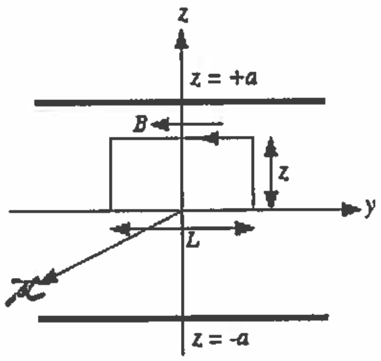
$$B = \frac{\mu_o i}{2\pi s}$$

Which is the same as we got using Biot-Savart's law but much simpler to use Ampere's law in this case.

Example 9:

A thick slab extending from $z = -a$ to $z = a$ carries a uniform volume current $\vec{J} = J\hat{\imath}$. Find the magnetic field both inside and outside the slab.

Because of the symmetry of the problem the magnetic field will be directed parallel to the y axis. The magnetic field in the region above the xy plane ($z > 0$) will be the mirror image of the field in the region below the xy plane ($z < 0$). The magnetic field in the xy plane ($z = 0$) will be equal to zero. Consider the Amperian loop shown in the figure. The current is flowing out of the paper, and we choice the direction of $d\vec{a}$ to be parallel to the direction of $\vec{J}$. Therefore,



$$\int \vec{J}.d\vec{a} = JzL \qquad 0 < z < a$$

$$\int \vec{J}.d\vec{a} = JaL \qquad z > a$$

The line integral of $\vec{B}$ is equal to:

$$\oint \vec{B}.d\vec{l} = BL = \mu_o \int \vec{J}.d\vec{a} = \mu_o JzL \qquad 0 < z < a$$

$$\vec{B} = -\mu_o Jz\hat{\jmath} \qquad 0 < z < a$$

$$\oint \vec{B}.d\vec{l} = BL = \mu_o \int \vec{J}.d\vec{a} = \mu_o JaL \qquad z > a$$

$$\vec{B} = -\mu_o Ja\hat{\jmath} \qquad z > a$$

$$\vec{B} = +\mu_o Ja\hat{\jmath} \qquad z < a$$

$\oint \bar{B}\cdot d\bar{l} = \int \cancel{\bar{B}}\cdot dl + \int \cancel{\bar{B}}\cdot d\bar{l} + \int \bar{B}\cdot d\bar{l} + \int \cancel{\bar{B}}\cdot d\bar{l}$

bottom $B=0$ — Right $\theta=90$ — top $= BL$ — left $\theta=90$

5.4. The Vector Potential

The magnetic field generated by a static current distribution is uniquely defined by the so-called **Maxwell equations for magnetostatics:**

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$$

Similarly, the electric field generated by a static charge distribution is uniquely defined by the so-called **Maxwell equations for electrostatics:**

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_o}$$

$$\vec{\nabla} \times \vec{E} = 0$$

The fact that the divergence of $\vec{B}$ is equal to zero suggests that there are no point charges for $\vec{B}$. Magnetic field lines therefore do not begin or end anywhere (in contrast to electric field lines that start on positive point charges and end on negative point charges).

Since a magnetic field is created by moving charges, a magnetic field can never be present without an electric field being present. In contrast, only an electric field will exist if the charges do not move.

Maxwell's equations for magnetostatics show that if the current density is known, both the divergence and the curl of the magnetic field are known. The Helmholtz theorem indicates that in that case there is a **vector potential** $\vec{A}$ such that:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

However, the vector potential is not uniquely defined. We can add to it the gradient of any scalar function f without changing its curl:

$$\vec{\nabla} \times \left(\vec{A} + \vec{\nabla} f\right) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla} f = \vec{\nabla} \times \vec{A}$$

The divergence of $\vec{A} + \vec{\nabla} f$ is equal to:

$$\vec{\nabla}.\left(\vec{A} + \vec{\nabla} f\right) = \vec{\nabla}.\vec{A} + \nabla^2 f$$

It turns out that we can always find a scalar function f such that the vector potential $\vec{A}$ is divergence-less. The main reason for imposing the requirement that $\vec{\nabla}.\vec{A} = 0$ is that it simplifies many equations involving the vector potential.

For example, Ampere's law rewritten in terms of $\vec{A}$ is:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2 \vec{A} = -\nabla^2 \vec{A} = \mu_o \vec{J}$$

$$\nabla^2 \vec{A} = -\mu_o \vec{J}$$

This equation is similar to Poisson's equation for a charge distribution ρ:

$$\nabla^2 V = -\frac{\rho}{\varepsilon_o}$$

Therefore, the vector potential $\vec{A}$ can be calculated from the current $\vec{J}$ in a manner similar to how we obtained V from ρ. Thus

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\Delta r} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau}{\Delta r} \text{ or } \frac{1}{4\pi\epsilon_0}\int \frac{\sigma\, da}{\Delta r} \text{ or } \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\, dl}{\Delta r}$$

$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{J}}{\Delta r} d\tau$ for a volume current

$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{K}}{\Delta r} da$ for a surface current

$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{I}}{\Delta r} dl = \frac{\mu_o I}{4\pi}\int \frac{d\vec{l}}{\Delta r}$ for a linear current

Note: these solutions require that the currents go to zero at infinity (similar to the requirement that ρ goes to zero at infinity). If the current does not got to zero at infinity then we will have to solve for $\vec{A}$ differently.

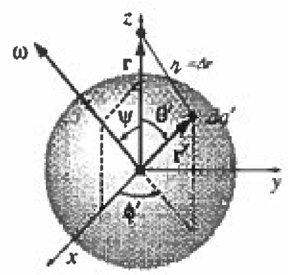
Example:

A spherical shell of radius R carrying a uniform surface charge σ, is set spinning at angular frequency ω, find the vector potential it produces at point $\vec{r}$.

Solution:

We will solve this problem by aligning $\vec{r}$ along z-axis and ω to be in the xz-plane, so that ω makes an angle ψ with the z-axis, as shown in the figure below.

The vector potential is given as:



$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{K}(\vec{r}')}{\Delta r} da'$$

Where $\vec{K} = \sigma\vec{v}$ and $\Delta r = \sqrt{R^2 + r^2 - 2rR\cos\theta'}$

$$da' = R^2 \sin\theta' \; d\theta' \; d\phi'$$

Now the velocity at $\vec{r}'$ in a rotating rigid body is given by $\vec{\omega} \times \vec{r}'$, so in this case:

$$\vec{v} = \vec{\omega} \times \vec{r}' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$\vec{v} = R\omega\left[(\sin\theta'\sin\phi'\cos\psi)\hat{\imath} + (\sin\theta'\cos\phi'\cos\psi - \cos\theta'\sin\psi)\hat{\jmath} + (\sin\theta'\sin\phi'\sin\psi)\hat{k}\right]$$

Notice that each term except one involves either $\cos\phi'$ or $\sin\phi'$ and their integration from 0 to 2π, is zero.

$$\int_0^{2\pi} \cos\phi' \, d\phi' = \int_0^{2\pi} \sin\phi' \, d\phi' = 0$$

Therefore,

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{K}(\vec{r}')}{\Delta r} da' = \frac{\mu_o}{4\pi}\int \frac{\sigma\vec{v}}{\Delta r} da' = \left(\frac{\mu_o\sigma R^3\omega\sin\psi}{2}\int \frac{-\cos(\theta')}{\sqrt{R^2 + r^2 - 2rR\cos\theta'}}\sin\theta' \, d\theta'\right)\hat{\jmath}$$

$A = \frac{\mu_0 \sigma R^3 \omega \sin\psi}{\cancel{r}} \times \frac{2R}{3r^2} = \frac{\mu_0 \sigma R^5}{3r^3} r\omega \sin\psi \quad r>R$

$$\int \frac{-\cos(\theta')}{\sqrt{R^2+r^2-2rR\cos\theta'}} \sin\theta' \, d\theta' = -\frac{1}{3R^2r^2}\left[(R^2+r^2+Rr)|R-r| - (R^2+r^2-Rr)(R+r)\right]$$

If the point r lies inside the sphere then r<R and the above expression would be equal to $(2r/3R^2)$ $\quad r<R$

and if r>R, then the above expression would be equal to $(2R/3r^2)$.

Also $\vec{\omega} \times \vec{r} = -\omega r \sin\psi \, \hat{j}$

So $\quad \hat{A} = \frac{\mu_0 \sigma R^{\cancel{2}} \omega \sin\psi}{\cancel{r}} \times \frac{2r}{3R^{\cancel{2}}} = \frac{\mu_0 \sigma R}{3} r\omega \sin\psi$

$$\vec{A} = \begin{cases} \dfrac{\mu_o R\sigma}{3}\,(\vec{\omega}\times\vec{r}) & \textit{for point inside the sphere} \quad r<R \\ \dfrac{\mu_o R^4\sigma}{3r^3}\,(\vec{\omega}\times\vec{r}) & \textit{for point outside the sphere} \quad r>R \end{cases}$$

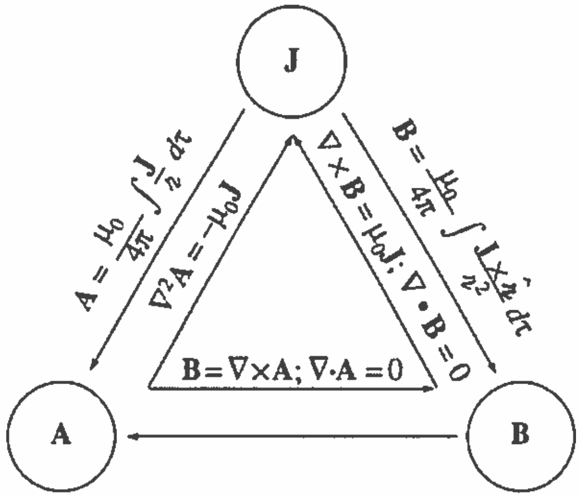
Now if we would like to align ω along z-axis and find the potential at r(r, θ, ϕ) in spherical coordinates, then:

$$\vec{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_o R\omega\sigma}{3} r\sin\theta\,\hat{\phi} & r<R \\ \dfrac{\mu_o R^4\omega\sigma}{3}\dfrac{\sin\theta}{r^2}\hat{\phi} & r>R \end{cases}$$

The magnetic field inside the spherical shell is uniform:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{2\mu_o R\omega\sigma}{3}\left(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) = \frac{2\mu_o R\omega\sigma}{3}\hat{z} = \frac{2\mu_o R\sigma}{3}\vec{\omega}$$

Magnetostatic Boundary Conditions:



Just as the electric field suffers a discontinuity at the surface charge similarly magnetic field suffers discontinuity at the surface current. But here the trangential component changes:

Since $\vec{\nabla}.\vec{B} = 0$

So $\oint \vec{B}.d\vec{a} = 0$

To a wafer thin pill-box straddling the surface:

$$B^{\perp}_{above} = B^{\perp}_{below}$$

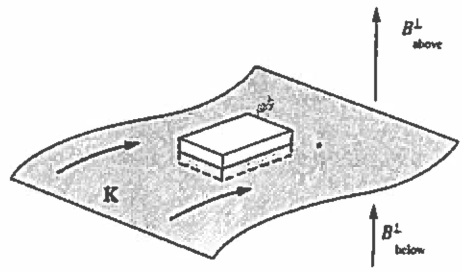
And for the tangential component:

$$\vec{\nabla}\chi\vec{B} = \mu_o\vec{J}$$

$$\oint \vec{B}.d\vec{l} = (B^{\parallel}_{above} - B^{\parallel}_{below})l = \mu_o I_{enclosed}$$

$$= \mu_o K l$$

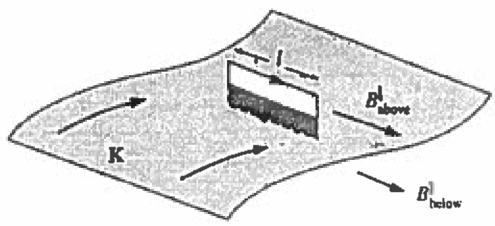
$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_o K$$



Or

$$\vec{B}_{above} - \vec{B}_{below} = \mu_o\vec{K} \times \hat{n}$$

Where $\hat{n}$ is a unit vector perpendicular to the surface, pointing upwards.



Like the scalar potential in electrostatics, the vector potential is continuous across any boundary:

$$\vec{A}_{above} = \vec{A}_{below}$$

$\vec{\nabla}.\vec{A} = 0$ guarantees that the normal component is continuous and the magnetic flux:

$$\Phi = \int \vec{B}.d\vec{a} = \int \vec{\nabla} \times \vec{A}.d\vec{a} = \oint \vec{A}.d\vec{l} = 0$$

Means that tangential components are continuous as well. But the derivative of $\vec{A}$ inherits the discontinuity of $\vec{B}$.

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{above}}{\partial n} = -\mu_o\vec{K}$$

The Multipole Expansion of the Magnetic Field

To calculate the vector potential of a localized current distribution at large distances we can use the multipole expansion by using the expansion of Δr.

$$\frac{1}{\Delta r} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

Consider a current loop with current I. The vector potential of this current loop can be written as:

$$\vec{A} = \frac{\mu_o I}{4\pi}\oint\frac{d\vec{l}'}{\Delta r} = \frac{\mu_o I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint(r')^n P_n(\cos\theta)d\vec{l}'$$

At large distance only the first couple of terms of the multipole expansion need to be considered:

$$\vec{A} = \frac{\mu_o I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r'\cos\theta'\, d\vec{l}' + \frac{1}{r^3}\oint r'^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)d\vec{l}' + \cdots\right]$$

The first term is called the **monopole term** and is equal to zero (since the line integral of $d\vec{l}'$ is equal to zero for any closed loop).

The second term, called the **dipole term**, is usually the dominant term. The vector potential generated by the dipole terms is equal to

$$\vec{A}_{dipole} = \frac{\mu_o I}{4\pi}\frac{1}{r^2}\oint r'\cos\theta'\, d\vec{l}' = \frac{\mu_o I}{4\pi}\frac{1}{r^2}\oint(\hat{r}.\vec{r}')d\vec{l}'$$

$$\oint(\hat{r}.\vec{r}')d\vec{l}' = -\frac{1}{2}\hat{r}\times\int\vec{r}'\times d\vec{l}' = \frac{\vec{m}}{I}\times\hat{r}$$

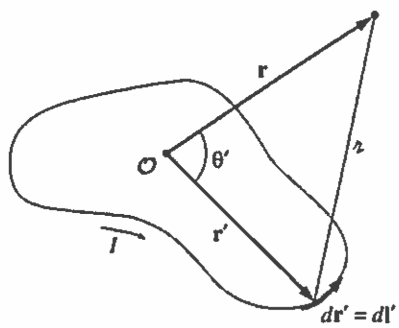
Where is the **magnetic dipole moment**, defined as:

$$\vec{m} = \frac{1}{2}I\oint\vec{r}'\times d\vec{l}'$$

If the current loop is a plane loop (current located on the surface of a plane) then $\frac{\int\vec{r}'\times d\vec{l}'}{2}$ is the area of the triangle shown in the figure below. Therefore,

$$\frac{\int\vec{r}'\times d\vec{l}'}{2} = a$$

where a is the area enclosed by the current loop. In this case, the dipole moment of the current loop is equal to



$$\vec{m} = I\vec{a}$$

where the direction of $\vec{a}$ must be consistent with the direction of the current in the loop (right-hand rule).

Assuming that the magnetic dipole is located at the origin of our coordinate system and that $\vec{m}$ is pointing along the positive z axis:

$$\vec{A}_{dipole} = \frac{\mu_o}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2} = \frac{\mu_o}{4\pi}\frac{m \sin\theta}{r^2}\hat{\phi}$$

The corresponding magnetic field is:

$$\vec{B}_{dipole} = \vec{\nabla} \times \vec{A}_{dipole} = \frac{1}{r sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\mu_o}{4\pi}\frac{m \sin\theta}{r^2}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\mu_o}{4\pi}\frac{m \sin\theta}{r^2}\right)\hat{\theta}$$

$$\vec{B}_{dipole} = \frac{\mu_o}{4\pi}\frac{m}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

The shape of the field generated by a magnetic dipole is identical to the shape of the electric field generated by an electric dipole.

Example 11:
A circular loop of wire, with radius *R*, lies in the *xy* plane, centered at the origin, and carries a current *I* running counterclockwise as viewed from the positive *z* axis.
a) What is its magnetic dipole moment?
b) What is its (approximate) magnetic field at points far from the origin?
c) Find magnetic field for points on the *z* axis.

Solution:
a)

$$\vec{m} = I\vec{a} = I * \pi R^2 \hat{k}$$

b) The magnetic field at large distances is approximately equal to

$$\vec{B}_{dipole} = \frac{\mu_o}{4\pi}\frac{m}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right] = \frac{\mu_o I R^2}{4r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$$

c) For points on the positive *z* axis $\theta = 0°$. Therefore, for $z>0$

$$\vec{B}_{dipole} = \frac{\mu_o I R^2}{2r^3}\hat{k}$$

Fore points on the negative *z* axis $\theta = 180°$. Therefore, for $z<0$

$$\vec{B}_{dipole} = -\frac{\mu_o I R^2}{2r^3}\hat{k}$$

The exact solution for $\vec{B}$ on the positive *z*-axis is

$$\vec{B}_{dipole} = -\frac{\mu_o I R^2}{2(R^2 + z^2)^{3/2}}\hat{k}$$

For $z \gg R$ the field is approximately equal to:

$$\vec{B}_{dipole} = -\frac{\mu_o I R^2}{2z^3}\hat{k}$$

which is consistent with the dipole field of the current loop.