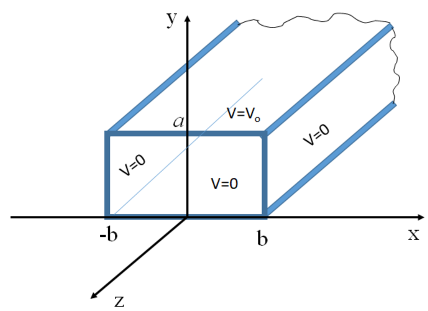
<u>**PHYS305 – Quiz#6**</u> Date: 07Nov2021

Name: ____ ID:__________________________

Q#1:

A section of a rectangular pipe is shown in the figure, suppose the bottom of the pipe and the two sides are grounded as shown in the figure. The potential on the top surface of the pipe at y=*a* is kept at potential V_o.



(a) What are the boundary conditions of this problem?
(b) Write the Laplace's equation.
(c) Use separation of variables and write general expression for the potential inside the pipe.
(d) Find the constants in the general solution by applying the appropriate boundary conditions.
(e) Write the expression for potential inside the pipe.

Solution:

a) The boundary conditions are:

i. $V = 0 \; at \; x = -b$
ii. $V = 0 \; at \; x = b$
iii. $V = 0 \; at \; y = 0$
iv. $V = V_o \; at \; y = a$

b) $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$

c) $V = X(x)Y(y)$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$\frac{1}{X}\frac{d^2 X}{dx^2} = C_1$ and $\frac{1}{Y}\frac{d^2 Y}{dy^2} = C_2$

$C_1 + C_2 = 0 \rightarrow C_2 = -C_1$

$$\frac{d^2 X(x)}{dx^2} = -k^2 X(x)$$

$$X(x) = A\sin(kx) + B\cos(kx)$$

$$\frac{d^2 Y(y)}{dy^2} = k^2 Y(y)$$

$$Y(y) = Ce^{ky} + De^{-ky}$$

$$V = X(x)Y(y) = (A\sin(kx) + B\cos(kx))\left(Ce^{ky} + De^{-ky}\right)$$

d) The solution along x-axis has to be symmetric due to its symmetric boundary conditions,so: A=0

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$$X(x) = B\cos(kx)$$

Using boundary codition (iii), $Y(y) = Ce^{ky} + De^{-ky}$

$Y(0) = C + D = 0 \rightarrow$ C=-D

$$Y(y) = C\left(e^{ky} - e^{-ky}\right) = C\sinh(ky)$$

$$V(x,y) = B\cos(kx)\sinh(ky)$$

Using boundary condition (ii) $V = 0\ at\ x = b$

$X(b) = B\cos(kb) = 0 \rightarrow k = \frac{n\pi}{2b}$ with $n = odd$

$$V(x,y) = B\cos\left(\frac{n\pi}{2b}x\right)\sinh\left(\frac{n\pi}{2b}y\right)$$

The general solution would be:

$$V(x,y) = \sum_{n=odd} B_n\cos\left(\frac{n\pi}{2b}x\right)\sinh\left(\frac{n\pi}{2b}y\right)$$

Using last boundary condition $V = V_o\ at\ y = a$

$$V(x,a) = V_o = \sum_{n=odd} B_n\cos\left(\frac{n\pi}{2b}x\right)\sinh\left(\frac{n\pi}{2b}a\right)$$

To find B_n, we can use Fourier's trick by multiplying both side by $\cos\left(\frac{n'\pi}{2b}x\right)$ and integrating from –b to +b:

$$\int_{-b}^{b} V_o\cos\left(\frac{n'\pi}{2b}x\right)dx = \sinh\left(\frac{n\pi}{2b}a\right)\sum_{n=odd} B_n\int_{-b}^{b}\cos\left(\frac{n\pi}{2b}x\right)\cos\left(\frac{n'\pi}{2b}x\right)dx$$

$$\int_{-b}^{b}\cos\left(\frac{n\pi}{2b}x\right)\cos\left(\frac{n'\pi}{2b}x\right)dx = \begin{cases} b\ if\ n' = n \\ 0\ if\ n' \neq n \end{cases}$$

$$\frac{1}{\frac{n\pi}{2b}}V_o\left[-\sin\left(\frac{n\pi}{2b}b\right) - \sin\left(\frac{n\pi}{2b}b\right)\right] = B_n\sinh\left(\frac{n\pi}{2b}a\right)b$$

$$B_n = -\frac{4V_o}{n\pi}\frac{(-1)^n}{\sinh\left(\frac{n\pi}{2b}a\right)}$$

$$V(x,y) = -\sum_{n=odd}\frac{4V_o}{n\pi}(-1)^n\frac{\sinh\left(\frac{n\pi}{2b}y\right)}{\sinh\left(\frac{n\pi}{2b}a\right)}\cos\left(\frac{n\pi}{2b}x\right)$$