Key

PHYS305 – Quiz#3

Date: 30Sep2021

Name: ______________________ ID:______________

Q#1:

(a) Compute the divergence of the following function:

$$\vec{v} = (r\cos\theta)\hat{r} + (r\sin\theta)\hat{\theta} + (r\sin\theta\cos\phi)\hat{\phi}$$

(b) Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy-plane and centered at the origin.

a) $\bar{\nabla}\cdot\bar{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(v_\phi)$

$= \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)$

$= 3\cos\theta + 2\cos\theta - \sin\phi \equiv 5\cos\theta - \sin\phi$

b) $\int(\bar{\nabla}\cdot\bar{v})\,d\tau = \oint\bar{v}\cdot d\bar{a} \longrightarrow$ (next page)

Q#2: Evaluate the following integral:

$$\int e^{-r}\left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right)d\tau$$

over the volume of a sphere of radius R, centered at the origin by two different methods.

by definition $\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(r)$

$\int e^{-r}\times 4\pi\delta^3(r)\,d\tau = 4\pi\times e^{-0} = 4\pi$

$\int\bar{\nabla}\cdot\left(e^{-r}\frac{\hat{r}}{r^2}\right)d\tau$

2nd method

$\int e^{-r}\times\bar{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = -\int\frac{\hat{r}}{r^2}\cdot\bar{\nabla}(e^{-r})\,d\tau + \oint e^{-r}\frac{\hat{r}}{r^2}\cdot d\bar{a}$

$\bar{\nabla}e^{-r} = \frac{\partial}{\partial r}(e^{-r})\,\hat{r} = -e^{-r}\hat{r}$; $da = r^2\sin\theta\, d\theta\, d\phi\,\hat{r}$

$= +\int_0^R\int_0^\pi\int_0^{2\pi}\frac{\hat{r}}{r^2}\cdot e^{-r}\hat{r}\times r^2\sin\theta\, dr\, d\theta\, d\phi + e^{-r}\times\int_0^\pi\int_0^{2\pi}\sin\theta\, d\theta\, d\phi\Big|_{r=R}$ $\longrightarrow$ next

Q#1 (b) $\int (\nabla \cdot \bar{v})\, d\tau = \oint \bar{v} \cdot d\bar{a}$

$$L.H.S = \int_0^R \int_0^{\pi/2} \int_0^{2\pi} (5\cos\theta - \sin\phi)\, r^2 \sin\theta \, d\theta \, d\phi \, dr = \frac{r^3}{3}\Big|_0^R \left[\int_0^{\pi/2}\int_0^{2\pi} 5\cos\theta \sin\theta \, d\theta \, d\phi - \int_0^{\pi/2}\int_0^{2\pi} \sin\theta \, d\theta \sin\phi \, d\phi \right]$$

let $\sin\theta = y$; $\cos\theta \, d\theta = dy$ when $\theta = 0, y = 0$ when $\theta = \frac{\pi}{2}, y = 1$

$$= \frac{R^3}{3}\left[5 \, \frac{y^2}{2}\Big|_0^1 \times 2\pi - \left(-\cos\theta\Big|_0^{\pi/2} \times -\cos\phi\Big|_0^{2\pi} \right) \right]$$

$$= \frac{R^3}{3}\left[5\pi - (-0+1)(-1+1) \right] = \boxed{\frac{5}{3}\pi R^3}$$

$$R.H.S = \oint \bar{v} \cdot d\bar{a} = \int_0^{\pi/2}\int_0^{2\pi} r\cos\theta \times r^2 \sin\theta \, d\theta \, d\phi \Big|_{r=R} + \int_0^R \int_0^{2\pi} r^2 \sin^2\theta \, dr \, d\phi \Big|_{\theta=\pi/2}$$

$\boxed{d\bar{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r} \quad dr \, r\sin\theta \, d\phi \, \hat{\theta}}$

$$= R^3 \times \frac{1}{2} \times 2\pi + \frac{r^3}{3}\Big|_0^R \times \sin^2\frac{\pi}{2} \times 2\pi$$

$$R.H.S = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3 = L.H.S$$

Q#2 Continued

$$= \frac{e^{-r}}{-1}\Big|_0^R \times 4\pi + e^{-R} \times 4\pi = -e^{-R} \times 4\pi + 4\pi + e^{-R} \times 4\pi$$

$$= 4\pi$$