Chapter 7. Electrodynamics

7.1. Electromotive Force

An electric current is flowing when the electric charges are in motion. In order to sustain an electric current we have to apply a force on these charges. In most materials, the current density $\vec{J}$ is proportional to the force per unit charge:

$$\vec{J} = \sigma \vec{f}$$

The constant of proportionality σ is called the **conductivity** of the material. Instead of specifying the conductivity, it is more common to specify the **resistivity** ρ:

$$\rho = \frac{1}{\sigma}$$

For conductors the resistivity is typically 10^{-8} Ω-m; for semiconductor it varies between 0.01 Ω-m and 1 Ω-m, and for insulators it varies between 10^5 Ω-m and 10^6 Ω-m. In most cases, the force on the charges is the electromagnetic force. In that case, the current density is equal to:

$$\vec{J} = \sigma\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

If the velocity of the charges is small the second term can be ignored, and the equation for $\vec{J}$ reduces to **Ohm's Law**:

$$\vec{J} = \sigma \vec{E}$$

Consider a wire of cross-sectional area A and length L. If a potential difference V is applied between the ends of the wire, it will produce an electric field inside the wire of magnitude:

$$E = \frac{V}{L}$$

The current density in the wire is therefore equal to

$$J = \sigma \frac{V}{L}$$

The total current flowing through the wire is therefore equal to

$$I = JA = \sigma A \frac{V}{L}$$

This equation shows that the current flowing from one electrode to the other electrode is proportional to the potential difference between them.

This is a rather surprising result since the charge carriers are constantly accelerating. However, the proportionality between the current and the potential difference has been found to be correct for most materials.

This relation can be written as:

$$V = IR$$

The constant of proportionality *R* is called the **resistance** of the material. It is in general a function of the geometry of the system and the conductivity of the materials between the electrodes. The unit of resistance is the **ohm** (Ω). The resistance of the wire is equal to

$$R = \frac{V}{I} = \frac{V}{\sigma A \frac{V}{L}} = \frac{1}{\sigma}\frac{L}{A} = \rho \frac{L}{A}$$

To create a current we have to perform work. The work required to move a unit charge across a potential difference *V* is equal to *V*. To establish a current *I*, we need to deliver a power *P*:

$$P = VI = I^2 R$$

The unit of power is **Watt** (1 W = 1 J/s). The work done by the electric force on the charge carriers is converted into heat (**Joule heating**).

Example 1:
Two concentric metal spherical shells, of radius *a* and *b*, respectively, are separated by weakly conducting material of conductivity σ.
a) If they are maintained at a potential difference *V*, what current flows from one to the other?
b) What is the resistance between the shells?

a) Suppose a charge *Q* is placed on the inner shell. The electric field in the region between the shells will be:

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\hat{r}$$

The corresponding potential difference between the spheres is equal to

$$V_a - V_b = -\int_a^b \vec{E}.d\vec{r} = \frac{Q}{4\pi\epsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Therefore, in order to maintain a potential difference *V* between the spheres, we must place a charge *Q* equal to

$$Q = \frac{4\pi\epsilon_o V}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

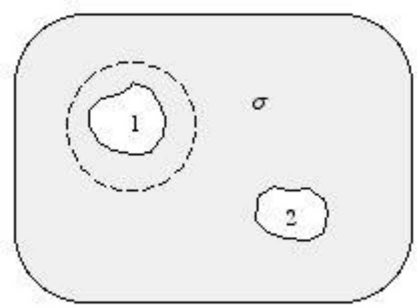
on the center shell. The total current flowing between the two shells is equal to

$$I = \oint \vec{J}.d\vec{a} = \sigma \oint \vec{E}.d\vec{a} = \sigma \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}\, 4\pi r^2 = \sigma \frac{Q}{\epsilon_o} = 4\pi\sigma \frac{V}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

b) The resistance between the shells can be obtained from Ohm's law:

$$R=\frac{V}{I}=\frac{V}{4\pi\sigma\frac{V}{\left(\frac{1}{a}-\frac{1}{b}\right)}}=\frac{1}{4\pi\sigma}\left(\frac{1}{a}-\frac{1}{b}\right)$$

Example 2:
a) Two metal objects are embedded in weakly conducting material of conductivity σ (see Figure below). Show that the resistance between them is related to the capacitance of the arrangement by $R=\frac{\varepsilon_o}{\sigma C}$



b) Suppose you connected a battery between 1 and 2 and charged them up to a potential difference V_0. If you then disconnect the battery, the charge will gradually leak off. Show that $V(t) = V_0 \exp(-t/\tau)$, and find the time constant τ in terms of ε_0 and σ.

a) Suppose a charge Q is placed on the positively charged conductor. The current flowing from the positively charged conductor is equal to

$$I=\oint\vec{J}.d\vec{a}$$

where the surface integral is taken over a surface that encloses the positively charged conductor (for example, the dashed surface around conductor 1). The expression for I can be rewritten in terms of the electric field as

$$I=\sigma\oint\vec{E}.d\vec{a}$$

Using Gauss's law to express the surface integral of $\vec{E}$ in terms of the total enclosed charge we obtain:

$$I=\sigma\frac{Q}{\varepsilon_o}$$

The charge on the conductor is related to the capacitance of the arrangement and the potential difference between the conductors:

$$Q=CV$$

The current I is therefore equal to:

$$I=\frac{\sigma}{\varepsilon_o}CV$$

The resistance of the system can be calculated using Ohm's law:

$$R = \frac{V}{I} = \frac{V}{\frac{\sigma}{\varepsilon_o} CV} = \frac{\varepsilon_o}{\sigma C}$$

b) The charge Q residing on the positively charged conductor is equal to

$$Q = CV = CRI = -CR\frac{dQ}{dt}$$

This equation can be rewritten as

$$\frac{dQ}{dt} + \frac{1}{CR}Q = 0$$

and has the following solution:

$$Q(t) = Q_o e^{-t/RC}$$

The potential difference V is equal to

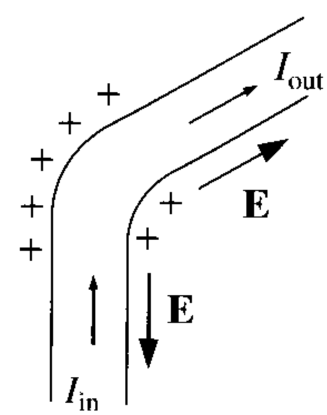
$$V(t) = \frac{Q(t)}{C} = \frac{Q_o}{C}e^{-t/RC} = V_o e^{-\frac{t}{RC}} = V_o e^{-\frac{t}{\tau}}$$

The decay constant τ is equal to

$$\tau = RC = \frac{\varepsilon_o}{\sigma}$$

In any electric circuit a current will only exist if a driving force is available. The most common sources of the driving force are batteries and generators. When a circuit is hooked up to a power source a current will start to flow. In a single-loop circuit the current will be the same everywhere.

Consider the situation in which the currents are not the same (see Figure below). If $I_{in} > I_{out}$ then positive charge will accumulate in the middle. This accumulation of positive charge will generate an electric field (see Figure below) that slows down the incoming charges and speeds up the outgoing charges.



A reduction in the velocity of the incoming charges will reduce the incoming current. An increase in the velocity of the outgoing charges will increase the outgoing current. The current will change until $I_{in} = I_{out}$.

The total force f on the charge carriers (per unit charge) is equal to the sum of the source force, f_s, and the electric force:

$$\vec{f} = \vec{f_s} + \vec{E}$$

The work required to move one unit of charge once around the circuit is equal to

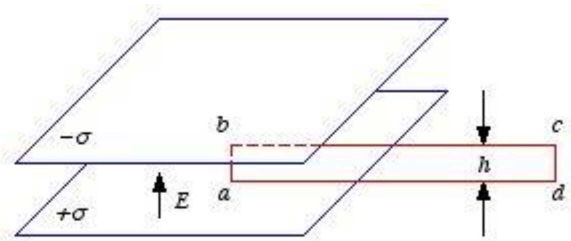
$$\oint \vec{f}.d\vec{l} = \oint \vec{f_s}.d\vec{l} + \oint \vec{E}.d\vec{l} = \oint \vec{f_s}.d\vec{l} = \varepsilon$$

where ε is called the **electromotive force** or **emf**. The emf determines the current flowing through the circuit. This can be most easily seen by rewriting the force $\vec{f}$ on the charge carriers in terms of the current density $\vec{J}$.

$$\varepsilon = \oint \vec{f}.d\vec{l} = \oint \frac{\vec{J}}{\sigma}.d\vec{l} = \oint \frac{I}{A\sigma} dl = I \oint \frac{dl}{A\sigma} = IR$$

Here, A is the cross-sectional area of the wire (perpendicular to the direction of the current).

Example 3:
a) Show that electrostatic force alone cannot be used to drive current around a circuit.
b) A rectangular loop of wire is situate so that one end is between the plates of a parallel-plate capacitor (see Figure below), oriented parallel to the field $E = \sigma/\varepsilon_0$. The other end is way outside, where the field is essentially zero. If the width of the loop is h and its total resistance is R, what current flows? Explain.



a) If only electrostatic forces are present then the force per unit charge is equal to the electrostatic force:

$$\vec{f} = \vec{E}$$

The associated emf is therefore equal to

$$\varepsilon = \oint \vec{f}.d\vec{l} = \oint \vec{E}.d\vec{l} = 0$$

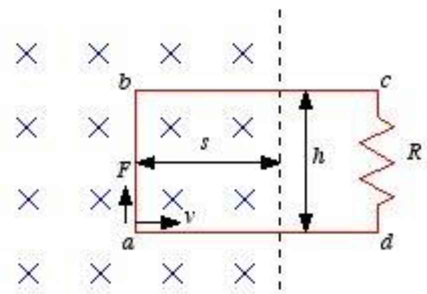
for any electrostatic field.
b) The only force on the charge carriers in the wire loop is the electric force. However, in part a) we concluded that the emf associate with an electric force, generated by an electrostatic field, is equal to zero. Therefore, the emf in the wire loop is equal to zero, and consequently the current in the loop is also equal to zero. Note: at first sight it might appear that there is a net emf, if we assume that the electric field generated by the capacitor is that of an ideal capacitor (that is a homogeneous field inside and no field outside). Under that assumption, the emf is equal to

$$\varepsilon = \int_a^b \vec{E}.d\vec{l} = \frac{\sigma}{\epsilon_o}h$$

The contribution of the path integral from *c* to *d* is equal to zero since the electric field is zero there, and the contribution of the path integrals between *b* and *c* and between *a* and *d* is equal to zero since the electric field and the displacement are perpendicular there.
Clearly the calculated emf is non-zero, and disagrees with the result of part a). The disagreement is a result of our incorrect assumption that the electric field outside the capacitor is equal to zero (there are fringing fields).

EMF Generated by Magnetic Field:
An important source of emf is the generator. In these devices, the EMF arises from the motion of a conducting wire through a magnetic field. Consider the system shown in the figure below where the magnetic field is only present in the region left of the dashed line. Consider the free charges on the conductor. Since it is moving with a velocity *v* in a magnetic field it will experience a magnetic force. The force on a positive charge *q* located in segment *ab* of the wire loop is equal to

$$F_o = qvB$$



The magnetic force per unit charge is therefore equal to

$$f_{mag} = \frac{F_o}{q} = vB$$

Since there are no other forces acting on the charges, the EMF generated will be entirely due to this magnetic force. The EMF will be equal to

$$\varepsilon = \oint \vec{f}_{mag}.d\vec{l} = \int_a^b \vec{f}_{mag}.d\vec{l} = vBh$$

The magnetic flux intercepted by the wire loop is equal to

$$\Phi = Bhs$$

The rate of change of the magnetic flux is equal to

$$\frac{d\Phi}{\mathrm{dt}} = Bh\frac{ds}{dt} = -Bhv$$

Comparing the rate of change of enclosed magnetic flux and the induced EMF we can conclude:

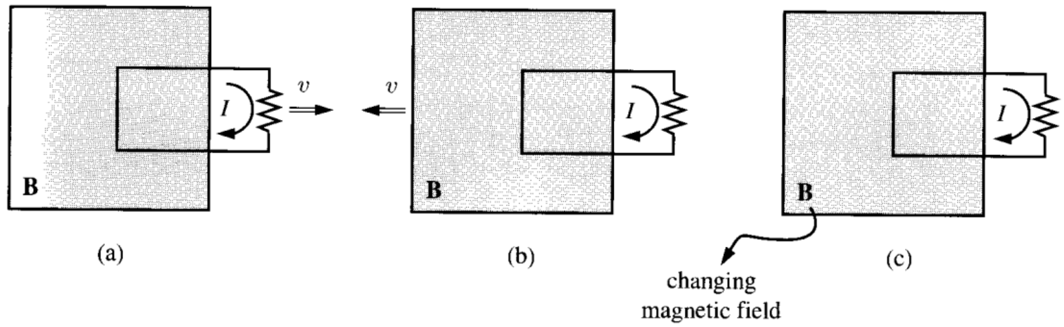
$$\varepsilon = -\frac{d\Phi}{\mathrm{dt}} = Bhv$$

This relation is called **the flux rule for motional emf**.

7.2. Faraday's Law

Faraday conducted three experiments as below:

1. He pulled a loop of wire to the right through a magnetic field and observed current flowed in the loop.
2. He pulled the magnet to the left and observed the current flowed in the loop of wire.
3. With both the loop and the magnet at rest, he changed the magnetic field and observed current flowed in the loop of wire.



In experiment 1 when the loop moves, it is the magnetic force that produces the EMF and it is called **motional EMF**.

$$\varepsilon = -\frac{d\Phi}{\mathrm{dt}}$$

In the second experiment, the loop is stationary and stationary charges do not experience magnetic force. In this case, the magnetic force does not play a role (since $v = 0$) and an electric field is responsible for the EMF. This electric field is not an electrostatic field (since electrostatic fields can not generate an EMF) but is induced by the changing magnetic field. The line integral of this electric field is:

$$\oint \vec{E}.d\vec{l} = \varepsilon = -\frac{d\Phi}{dt}$$

This equation can be rewritten by applying Stoke's theorem:

$$\oint \vec{E}.d\vec{l} = \int (\vec{\nabla} \times \vec{E}).d\vec{a} = -\frac{d}{dt}\int \vec{B}.d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t}.d\vec{a}$$

Since we have not made any assumption about the surface, this equation can only be true if

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

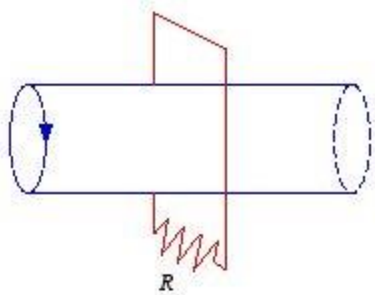
This relation is called **Faraday's law in differential form**. This is differential form of Faraday's law and it reduces to $\vec{\nabla} \times \vec{E} = 0$ for a constant magnetic field.
In the third experiment, the magnetic field changes without moving the loop or the magnet, but according to Faraday's law electric field will be induced giving rise to EMF.

We can put all three cases as whenever the magnetic flux changes in a loop of wire for whatever reason there will be an EMF produced:

$$\varepsilon = -\frac{d\Phi}{dt}$$

The direction of the currents generated by the changing magnetic field can be obtained most easily using **Lenz's law** which states that *"If a current flows, it will be in such a direction that the magnetic field it produces tends to counteract the change in flux that induced the EMF."*

Example 4:
A long solenoid of radius *a*, carrying *n* turns per unit length, is looped by a wire of resistance *R* (see Figure below).
a) If the current in the solenoid is increasing, $\frac{dI}{dt} = k = constant$, what current flows in the loop, and which way (left or right) does it pass through the resistor.
b) If the current *I* in the solenoid is constant but the solenoid is pulled out of the loop and reinserted in the opposite direction what total charge passes through the resistor?



a) Assume that the solenoid is an ideal solenoid; that is

$$\vec{B} = \mu_o n I \hat{k}$$

If the current in the solenoid increases, the strength of the magnetic field also increases. The rate of change in the strength of the magnetic field is equal to

$$\frac{d\vec{B}}{dt} = \mu_o n \frac{dI}{dt}\hat{k} = \mu_o n k \hat{k}$$

The magnetic flux intercepted by the wire loop is equal to

$$\Phi = \pi a^2 B$$

The corresponding rate of change of the magnetic flux is equal to

$$\frac{\mathrm{d}\Phi}{\mathrm{dt}} = \pi a^2 \frac{dB}{dt} = \pi a^2 \mu_o n k$$

The induced EMF can be obtained from the flux law:

$$\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{dt}} = -\pi a^2 \mu_o n k$$

The current induced in the wire loop is equal to

$$I = \frac{\varepsilon}{R} = \frac{\pi a^2}{R} \mu_o n k$$

The solenoidal magnetic field points from left to right. An increase in the strength of the magnetic field will induce a current in the loop directed such that the magnetic field it produces point from right to left (Lenz's law). Therefore, the current flows from left to right through the resistor.

b) The change in the magnetic flux enclosed by the wire loop is equal to

$$\Delta\Phi = \pi a^2 \Delta B = 2\pi a^2 \mu_o n I$$

The current flowing through the resistor is equal to

$$I = \frac{\varepsilon}{R} = \frac{1}{R}\frac{d\Phi}{dt} = \frac{dQ}{dt}$$

$$\Delta Q = -\frac{1}{R}\int \frac{d\Phi}{dt} dt = -\frac{\Delta\Phi}{R}$$

$$\Delta Q == -\frac{\Delta\Phi}{R} = -\frac{2\pi a^2 \mu_o n I}{R}$$

7.2.2 The Induced Electric Field

Faraday's discovery found that there are two kind of electric fields, one due to static charges and the other due to changing magnetic field.

Electric field due to static charges can be calculated using Coulomb's law but the electric field due to changing magnetic field can be calculated by exploiting the analogy between Faraday's law and Ampere's law:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{and} \quad \vec{\nabla} \times \vec{B} = -\mu_o \vec{J}$$

Since Curl alone is not enough to determine a field so we need divergence as well. But as long as $\vec{E}$ is purely Faraday field and in the absence of any charges, Gauss's law says:

$\vec{\nabla}.\vec{E} = 0$ and for magnetic field $\vec{\nabla}.\vec{B} = 0$

Hence the parallel is complete, so we can conclude that Faraday induced electric fields can be determined by $-\left(\frac{\partial \vec{B}}{\partial t}\right)$ in exactly the same as we determine magnetostatic field by $\mu_o \vec{J}$.

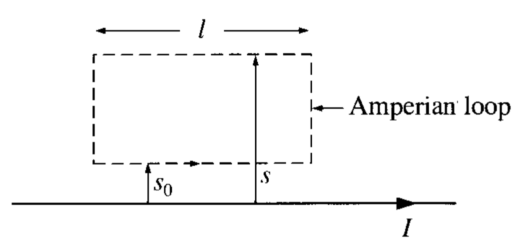
If symmetry permits we can use all the tricks of Ampere's law to determine the electric field, like

$$\oint \vec{B}.d\vec{l} = \mu_o I_{enc}$$

$$\oint \vec{E}.d\vec{l} = -\frac{d\Phi}{dt}$$

Example 5:

An infinitely long straight wire carries a slowly varying current I(t). Determine the induced electric field as a function of distance s from the wire.



Solution:

In a quasistatic approximation, the magnetic field produced by the current at distance s is:

$$B = \frac{\mu_o I}{2\pi s}$$

For the rectangular Amperian loop shown in the figure:

$$\oint \vec{E}.d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}.d\vec{a} = -\int \frac{\mu_o}{2\pi s}\frac{dI}{dt}\, da$$

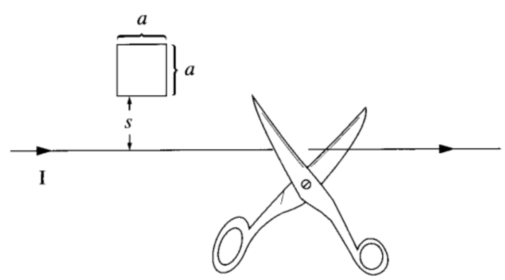
$$E(s_o)l - E(s)l = -\frac{\mu_o}{2\pi}\frac{dI}{dt}\int_{s_o}^{s}\frac{l}{s}\, ds = -\frac{\mu_o l}{2\pi}\frac{dI}{dt}[\ln(s) - \ln(s_o)]$$

$$E(s) = \frac{\mu_o}{2\pi}\frac{dI}{dt}\ln(s) + K$$

Example 6:

A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I as shown in the figure. Now someone cuts the wire, so that I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, then turn the current down gradually:

$$I(t) = \begin{cases}(1-\alpha t)I & for\ 0 \le t \le 1/\alpha \\ 0 & for\ t \ge 1/\alpha\end{cases}$$



Solution:

$$\Phi = \int \vec{B}.d\vec{a} = \frac{\mu_o I a}{2\pi}\int_{s}^{s+a}\frac{ds'}{s'} = \frac{\mu_o I a}{2\pi}\ln\left(\frac{s+a}{s}\right)$$

$$\varepsilon = I_{loop}R = \frac{dQ}{dt}R = -\frac{d\Phi}{dt} = -\frac{\mu_o a}{2\pi}\ln\left(\frac{s+a}{s}\right)\frac{dI}{dt}$$

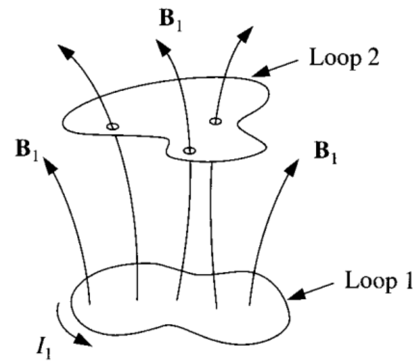
$$dQ = -\frac{\mu_o a}{2\pi R}\ln\left(\frac{s+a}{s}\right)dI$$

$$Q = -\frac{\mu_o I a}{2\pi R}\ln\left(\frac{s+a}{s}\right)$$

7.3. Inductance

Consider two loops: loop 1 and loop 2 as shown in the figure below.
A current I_1 flowing through loop 1 will produce a magnetic field at the position of loop 2 equal to

$$\vec{B}_1 = \frac{\mu_o I_1}{4\pi}\oint\frac{d\vec{l}_1\times\Delta\hat{r}}{\Delta r^2}$$



The magnetic field generated by loop is directly proportional to the current in the loop I_1. Hence

The magnetic flux through loop 2 is equal to

$$\Phi_2 = \int \vec{B}_1 . d\vec{a}_2 = \int(\vec{\nabla}\times\vec{A}_1). d\vec{a}_2 = \oint \vec{A}_1 . d\vec{l}_2$$

$$\vec{A}_1 = \frac{\mu_o I_1}{4\pi}\oint\frac{d\vec{l}_1}{\Delta\vec{r}}$$

$$\Phi_2 = \frac{\mu_o I_1}{4\pi}\oint\left(\oint\frac{d\vec{l}_1}{\Delta\vec{r}}\right). d\vec{l}_2 = M_{21}I_1$$

$$M_{21} = \frac{\mu_o}{4\pi}\oint\left(\oint\frac{d\vec{l}_1}{\Delta\vec{r}}\right). d\vec{l}_2$$

Here, M_{21} is called the **mutual inductance** of the two loops. It is a purely geometrical quantity that depends on the sizes, shapes and relative positions of the two loops. It does not change if we switch the role of loop 1 and loop 2: the flux through loop 2 when we run a current I around loop 1 is exactly the same as the flux through loop 1 when we send the same current I around loop 2.
Besides inducing an EMF in a nearby loop, the changing current in loop 1 also induces an EMF in loop 1.
The flux through loop 1 generated by the current in loop 1 is equal to

$$\Phi_1 = LI_1$$

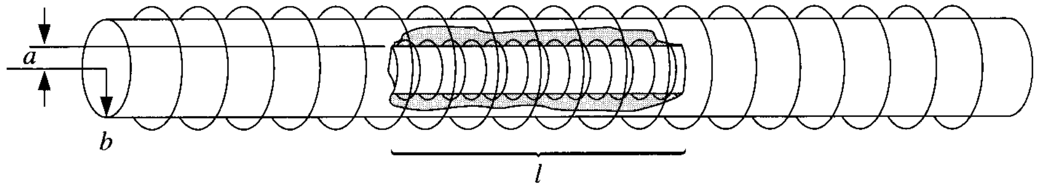
The constant of proportionality is called the **self-inductance** (or just **Inductance**). The unit of inductance is the **Henrie** (*H*). Henry is volt-second/meter.

As with M it depends on the geometry and size of the loop. If the current changes EMF induced in the loop is:

$$\varepsilon = -L\frac{dI}{dt}$$

Suppose a current I is flowing around a loop when suddenly wire is cut and the current drops instantaneously to zero. This generates a back EMF, for although I may be small but $\left(\frac{dI}{dt}\right)$ is enormous. That's why we often see a spark when we suddenly unplug the iron or a toaster- electromagnetic induction is desperately trying to keep the current going, it can even jump the gap in the circuit. But nothing so dramatic happens when we plug-in the iron or the toaster because induction opposes the sudden increase in the current, making instead a smooth and continuous build-up.

Example 7:
A short solenoid of length l and radius a, with n_1 turns per unit length lies on the axis of a very long solenoid of radius b with n_2 turns per unit length as shown below. Current I flows in the short solenoid. What is the flux through the long solenoid.



Solution: Since the inner solenoid is short it has very complicated field and the flux through each ring of the bigger solenoid will be different. Instead we will exploit the equality of the mutual inductances and use the magnetic field of the bigger solenoid and find the flux through the shorter solenoid and they both should be same.

The magnetic field of the bigger solenoid is constant in the region of smaller solenoid:

$$B = \mu_o n_2 I$$

The flux through the single loop of a smaller solenoid will be:

$$\Phi = B\pi a^2 = \mu_o n_2 I \pi a^2$$

Since there are $n_1 l$ turns in total then the flux through the whole solenoid would be:

$$\Phi = \mu_o n_1 n_2 l \pi a^2 I$$

$$M = \mu_o n_1 n_2 l \pi a^2$$

If the current varies in loop 1, it will induce EMF in loop 2

$$\varepsilon_2 = -\frac{d\Phi_2}{\mathrm{dt}} = -M\frac{dI_1}{dt}$$

And this changing current in loop 1 will not only induce EMF in loop 2 but also induce EMF in loop 1 itself. And the flux through loop 1 is proportional to the current in loop 1.

$$\Phi = \mathrm{LI}$$

Where L is the **self-Inductance** and the EMF induced in loop 1 due to change in current in loop 1, will be:

$$\varepsilon_1 = -\frac{d\Phi_1}{\mathrm{dt}} = -L\frac{dI_1}{dt}$$

Example 8:
Find the self-inductance per unit length of a long solenoid, of radius R, carrying *n* turns per unit length.

Solution:

$$B = \mu_o n I$$
$$\Phi = \mu_o n I \pi R^2$$

In a length *l* there are *nl* turns so the flux through all these turns will be:

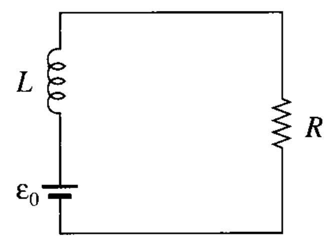
$$\Phi = \mu_o n^2 l \pi R^2 I = LI$$

$$\frac{L}{l} = \mu_o n^2 \pi R^2$$

Example 9:

Suppose for instance, that a battery (which supplies a constant EMF ε_o) is connected to a circuit of resistance R and inductance L. What current flows?

Solution:



The total EMF in the circuit is that provided by the battery plus the one resulting from the self-inductance.

$$\varepsilon_o - L\frac{dI}{dt} = IR$$

$$\frac{L}{R}\frac{dI}{dt} + I = \frac{\varepsilon_o}{R}$$

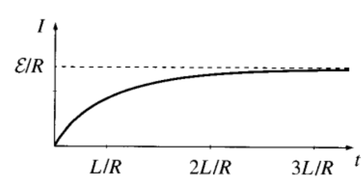
The solution to this differential equation is:

$$I(t) = \frac{\varepsilon_o}{R} + ke^{-(R/L)t}$$

Where k is a constant to be determined by the initial conditions. If the circuit is plugged in at time t=0 (so $I(0) = 0$), then k has the value $-\varepsilon_o/R$, and

$$I(t) = \frac{\varepsilon_o}{R}\left[1 - e^{-\left(\frac{R}{L}\right)t}\right]$$

If there was no inductance in the circuit then current would immediately jump to $\frac{\varepsilon_o}{R}$.



The quantity $\tau = \frac{L}{R}$ is called the time constant and it is 2//3rd of the time it takes to reach maximum value.

7.2.4 Energy in Magnetic Fields

It takes a certain amount of energy to start a current flowing in a circuit. The reason energy is required because work needs to be done against the back EMF to get the current going. This is a fixed amount and it is recovered when the current is turned OFF. In the meantime, it represent the energy latent in the circuit or energy stored in a magnetic field.

The work done on a unit charge, against back EMF, in one trip around the circuit is $-\varepsilon$ (here minus sign indicates the work is not done by the EMF). Since the amount of charge passing down the wire per unit time is I, so the total work done per unit time is:

$$\frac{dW}{dt} = -\varepsilon I = LI\frac{dI}{dt}$$

If we start with the zero current and build it up to I, then the total work done is:

$$W = \frac{1}{2}LI^2$$

It does not depend on how long does it take to reach the current I but only on the geometry of the loop (interms of L) and the final current I.

Alternate way to represent W:

Since the flux through the loop is:

$$\Phi = LI$$

Also

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = \int_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{A} \cdot d\mathbf{l},$$

where P is the perimeter of the loop and S is any surface bounded by P.

$$LI = \oint \vec{A}.d\vec{l}$$

Therefore

$$W = \frac{1}{2}LI^2 = \frac{1}{2}I\oint \vec{A}.d\vec{l} = \frac{1}{2}\oint (\vec{A}.\vec{I})dl$$

If there is a volume current, the above equation can be generalized as:

$$W = \frac{1}{2}\int_V (\vec{A}.\vec{J})d\tau$$

Using Ampere's law $\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$:

$$W = \frac{1}{2\mu_o} \int_V \vec{A}.\left(\vec{\nabla} \times \vec{B}\right) d\tau$$

$$\vec{\nabla}.\left(\vec{A} \times \vec{B}\right) = \vec{B}.\left(\vec{\nabla} \times \vec{A}\right) - \vec{A}.\left(\vec{\nabla} \times \vec{B}\right)$$

$$\vec{A}.\left(\vec{\nabla} \times \vec{B}\right) = \vec{B}.\left(\vec{\nabla} \times \vec{A}\right) - \vec{\nabla}.\left(\vec{A} \times \vec{B}\right) = B^2 - \vec{\nabla}.\left(\vec{A} \times \vec{B}\right)$$

$$W = \frac{1}{2\mu_o} \left[\int_V B^2 d\tau - \int_V \vec{\nabla}.\left(\vec{A} \times \vec{B}\right) d\tau \right]$$

$$W = \frac{1}{2\mu_o} \left[\int_V B^2 d\tau - \int_S \left(\vec{A} \times \vec{B}\right). d\vec{a} \right]$$

Where S is the surface bounding the volume V. And Integration is over the entire volume occupied by the current. But any region larger than this will also be OK because J would be zero everywhere except at the source. For larger region we pick greater contribution from the volume integral and negligible from the surface integral, so we can ignore the surface integral part:

Hence

$$W = \frac{1}{2\mu_o} \int_{all\ space} B^2 d\tau$$

From this result we can say that energy is stored in the magnetic field in the amount $\left(\frac{B^2}{2\mu_o}\right)$ per unit volume.

The main point is that producing a magnetic field, where previously there was none, requires changing the field, and a changing magnetic field according to Faraday, induces an electric field. The latter can do the work.

In the beginning there is no E and at the end there is no E, but in between when B is building up, there is an induced E, and it is against this that the work is done.

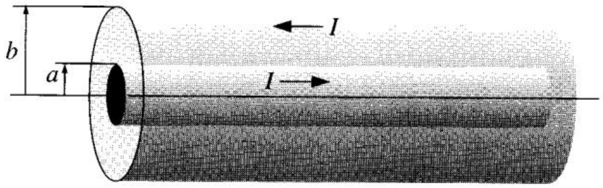
Remember energy stored in electrostatic field was:

$$W_{elec} = \frac{1}{2} \int (V\rho) d\tau = \frac{\epsilon_o}{2} \int_{all\ space} E^2 d\tau$$

$$W_{mag} = W = \frac{1}{2} \int_V \left(\vec{A}.\vec{J}\right) d\tau = \frac{1}{2\mu_o} \int_{all\ space} B^2 d\tau$$

Example 10:

A long coaxial cable carries current I (the current flows down the surface of the inner cylinder of radius a, and back along the outer cylinder of radius b) as shown in the figure below. Find the magnetic energy stored in a section of length *l*.



Solution:

Since the current in the inner cylinder is I, so using Ampere's law we can find the magnetic field in the region between the inner cylinder and the outer cylinder, as:

$$\oint \vec{B}\,.\,d\vec{l} = \mu_o I$$

$$B * 2\pi s = \mu_o I \;\rightarrow B = \frac{\mu_o I}{2\pi s}$$

[magnetic field will be zero outside the bigger cylinder because net current enclosed will be zero. B inside the inner cylinder is also zero because inner cylinder has only surface current.]

The energy per unit volume is:

$$\frac{W}{Volume} = \frac{B^2}{2\mu_o} = \frac{\left(\frac{\mu_o I}{2\pi s}\right)^2}{2\mu_o} = \frac{\mu_o I^2}{8\pi^2 s^2}$$

Energy stored in the cylindrical shell of radius $s > a$ and length *l* and thickness *ds* is given by:

$$W = \frac{\mu_o I^2}{8\pi^2 s^2} * 2\pi s l * ds = \frac{\mu_o I^2 l}{4\pi}\left(\frac{ds}{s}\right)$$

Integrating from *a* to *b*, we get:

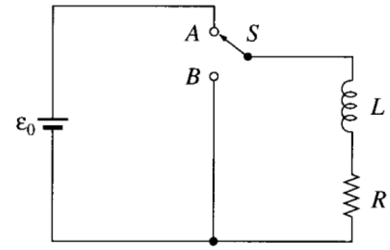
$$W = \int_a^b \frac{\mu_o I^2 l}{4\pi}\left(\frac{ds}{s}\right) = \frac{\mu_o I^2 l}{4\pi}\ln\left(\frac{b}{a}\right)$$

Since also, $W = \frac{1}{2} L I^2$, so

$$L = \frac{\mu_o l}{2\pi}\ln\left(\frac{b}{a}\right)$$

Example 11:

Suppose the circuit in the figure below has been connected for a very long time when suddenly at t=0, the switch is thrown at B, disconnecting the battery from the circuit.



a) What is the current I at any subsequent time.
b) What is the total energy delivered to the resistor.
c) Show that this is energy is same as originally stored in the inductor.

Solution:

(a) Initial current is $I_o = \frac{\varepsilon_o}{R}$

$$L\frac{dI}{dt} = IR$$

$$I = I_o e^{-\frac{R}{L}t} = \frac{\varepsilon_o}{R} e^{-\left(\frac{R}{L}\right)t}$$

(b)

$$P = I^2 R = \frac{\varepsilon_o^2}{R} e^{-2\left(\frac{R}{L}\right)t} = \frac{dW}{dt}$$

$$W = \frac{\varepsilon_o^2}{R}\int_0^\infty e^{-2\left(\frac{R}{L}\right)t}dt = \frac{\varepsilon_o^2}{R}\left(-\frac{L}{2R}e^{-\frac{2R}{L}t}\right)\Big|_0^\infty = \frac{1}{2}L\left(\frac{\varepsilon_o}{R}\right)^2$$

(c)

$$W = \frac{1}{2}LI_o^2 = \frac{1}{2}L\left(\frac{\varepsilon_o}{R}\right)^2$$

7.4. The Maxwell Equations

The electric and magnetic fields in electrostatics and magnetostatics are described by the following four equations:

$$\overline{\nabla} \bullet \overline{E} = \frac{1}{\varepsilon_0}\rho \qquad \text{Gauss' Law}$$

$$\overline{\nabla} \bullet \overline{B} = 0$$

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \text{Faraday's Law}$$

$$\overline{\nabla} \times \overline{B} = \mu_0 \overline{J} \qquad \text{Ampere's Law}$$

In systems with non-steady currents not all of these equations are valid anymore. For example,

$$\overline{\nabla} \bullet \left(\overline{\nabla} \times \overline{B}\right) = 0$$

for every vector function. However, according to Ampere's law

$$\overline{\nabla} \bullet \left(\overline{\nabla} \times \overline{B}\right) = \mu_0\left(\overline{\nabla} \bullet \overline{J}\right)$$

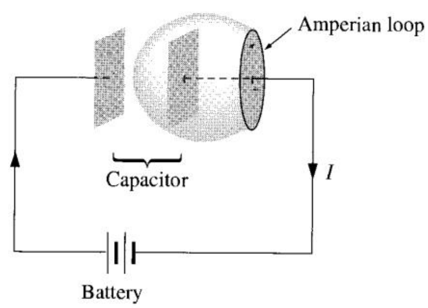
which is only zero for steady currents (for which $\overline{J}$ is a constant, independent of position). For non-steady currents

$$\mu_0\left(\overline{\nabla} \bullet \overline{J}\right) \neq 0 = \overline{\nabla} \bullet \left(\overline{\nabla} \times \overline{B}\right)$$

We thus conclude that Ampere's law does not hold for non-steady currents. The failure of Ampere's law can also be observed in a system in which a capacitor is being charged (see Figure 7.8). During the charging process a current *I* is flowing through the wire, and consequently there will be a magnetic field present. The magnetic field generated by the charging current can be calculated using Ampere's law. When we are far away from the capacitor the generated magnetic field will be that of a line current. Consider an Amperian loop of radius *r*, centered on the wire. The line integral of $\overline{B}$ around this loop is equal to

$$\oint \overline{B} \bullet d\overline{l} = 2\pi r B$$

According to Ampere's law the line integral of $\overline{B}$ around a closed loop is proportional to the current intercepted by a surface spanned by this loop. For the system shown in the figure, the intercepted current is ill defined. Consider first surface 1. The current intercepted by surface 1 is equal to *I*. Surface 2 is also spanned by the Amperian loop, but the current intercepted by this loop is zero. We thus conclude that Ampere's law does not apply in systems where the current is not continuous.



Maxwell modified Ampere's law in the following manner:

$$\overline{\nabla} \times \overline{B} = \mu_0 \left(\overline{J} + \varepsilon_0 \frac{\partial \overline{E}}{\partial t} \right)$$

The term added by Maxwell is called the **displacement current**. It is defined as

$$\overline{J}_d = \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$$

Consider the region between the capacitor plates in Figure 7.8. The electric field in this region is equal to

$$\overline{E} = \frac{\sigma}{\varepsilon_0} \hat{k} = \frac{Q}{\varepsilon_0 A} \hat{k}$$

where we have assumed that the field produced is that of an ideal capacitor with surface area *A* and the *z* axis is in the direction of the current. The rate of change of the electric field is equal to

$$\frac{\partial \overline{E}}{\partial t} = \frac{1}{\varepsilon_0} \frac{\partial \sigma}{\partial t} \hat{k} = \frac{1}{\varepsilon_0 A} \frac{\partial Q}{\partial t} \hat{k} = \frac{I}{\varepsilon_0 A} \hat{k}$$

The surface integral of $\partial \overline{E} / \partial t$ across surface 2 is therefore equal to

$$\oint \frac{\partial \overline{E}}{\partial t} \bullet d\overline{a} = \frac{I}{\varepsilon_0}$$

The surface integral of $\overline{\nabla} \times \overline{B}$ across surface 2 is equal to

$$\oint_{\text{Surface 2}} \left(\overline{\nabla} \times \overline{B} \right) \bullet d\overline{a} = \mu_0 \oint_{\text{Surface 2}} \overline{J} \bullet d\overline{a} + \mu_0 \varepsilon_0 \oint_{\text{Surface 2}} \frac{\partial \overline{E}}{\partial t} \bullet d\overline{a} = \mu_0 I$$

The modification of Ampere's law by Maxwell insures that the surface integral of $\overline{\nabla} \times \overline{B}$ is independent of the surface chosen. In electrostatics and magnetostatics the electric and magnetic fields are constant in time, and therefore, the new form of Ampere's law reduces to the form of Ampere's law we have been using so far.

In a region where there are no free charges or free currents Maxwell's equations become very symmetric

$$\overline{\nabla} \bullet \overline{E} = 0 \qquad \overline{\nabla} \bullet \overline{B} = 0$$

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \overline{\nabla} \times \overline{B} = \mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$$

The symmetry is broken when electric charges are present, unless besides electric charges there are **magnetic monopoles**. If the magnetic charge density is equal to η and the magnetic current is equal to $\bar{K}$ then Maxwell's equation become

$$\bar{\nabla} \bullet \bar{E} = \frac{\rho}{\varepsilon_0} \qquad \bar{\nabla} \bullet \bar{B} = \mu_0 \eta$$

$$\bar{\nabla} \times \bar{E} = -\mu_0 \bar{K} - \frac{\partial \bar{B}}{\partial t} \qquad \bar{\nabla} \times \bar{B} = \mu_0 \bar{J} + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$$

To obtain Maxwell's equations that describe the electric and magnetic fields in matter we must take the bound charges and bound currents into account:

$$\rho_b = -\bar{\nabla} \bullet \bar{P}$$

$$\bar{J}_b = \bar{\nabla} \times \bar{M}$$

In the non-static case, the polarization can be time dependent. Therefore, also the bound charge density is time dependent, and a net current can be associated with the change in the bound charge density. This current is called the **polarization current** $\bar{J}_p$ and is equal to

$$\bar{J}_p = \frac{\partial \bar{P}}{\partial t}$$

Maxwell's equations in matter are therefore equal to

$$\bar{\nabla} \bullet \bar{E} = \frac{1}{\varepsilon_0}\left(\rho_f + \rho_b\right) = \frac{1}{\varepsilon_0}\left(\rho_f - \bar{\nabla} \bullet \bar{P}\right) \qquad \text{Gauss' Law}$$

$$\bar{\nabla} \bullet \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \qquad \text{Faraday's Law}$$

$$\bar{\nabla} \times \bar{B} = \mu_0\left(\bar{J}_f + \bar{J}_b + \bar{J}_p\right) + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t} = \mu_0\left(\bar{J}_f + \bar{\nabla} \times \bar{M} + \frac{\partial \bar{P}}{\partial t}\right) + \mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t} \qquad \text{Ampere's Law}$$

It is common to rewrite Maxwell's equations in terms of the parameters we can control (the free charge density and the free current density). Gauss's law can be rewritten as

$$\bar{\nabla} \bullet \left(\varepsilon_0 \bar{E} + \bar{P}\right) = \bar{\nabla} \bullet \bar{D} = \rho_f$$

where $\bar{D}$ is called the **electric displacement**. Ampere's law can be rewritten as

$$\bar{\nabla} \times \left(\frac{\bar{B}}{\mu_0} - \bar{M}\right) = \bar{\nabla} \times \bar{H} = \bar{J}_f + \frac{\partial}{\partial t}\left(\varepsilon_0 \bar{E} + \bar{P}\right) = \bar{J}_f + \frac{\partial \bar{D}}{\partial t}$$

where $\bar{H}$ is called the ***H* field**. The most general form of Maxwell's equations, in terms of the free charges and free currents, is given by

$$\overline{\nabla} \bullet \overline{D} = \rho_f \qquad \overline{\nabla} \bullet \overline{B} = 0$$

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \overline{\nabla} \times \overline{H} = \overline{J}_f + \frac{\partial \overline{D}}{\partial t}$$