Chapter 8. Conservation Laws

8.1 Charge and Energy

8.1.1 The Continuity Equation

In this chapter we study the conservation of energy, momentum and angular momentum in electrodynamics. They all start from the conservation of charge.

The conservation of charge means that if the total charge in some volume changes then exactly that amount of charge must have passed in or out through the surface.

$$Q(t) = \int_V \rho(\vec{r}, t) d\tau$$

And the current flowing through the boundary is:

$$\frac{dQ}{dt} = -\int_S \vec{J} . d\vec{a}$$

Combining the above two equations, we get

$$\int_V \frac{\partial \rho}{\partial t} d\tau = -\int_V \vec{\nabla} . \vec{J} \, d\tau$$

Since this is true for any volume, so

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} . \vec{J}$$

This is called continuity equation, the precise mathematical statement of conservation of charge. In this chapter we will learn energy density and momentum density (analogs to ρ) and energy current and momentum current (analogs to $\vec{J}$).

8.1.2 Poynting's Theorem

Since we know the work necessary to assemble static charge distribution is:

$$W_e = \frac{\epsilon_o}{2} \int E^2 d\tau$$

Where **E** is the resulting electric field.

Similarly the work required to get current going (against the back EMF) is:

$$W_m = \frac{1}{2\mu_o} \int B^2 d\tau$$

Where **B** is the resulting magnetic field.

Hence the total energy stored in an electromagnetic field is given by:

$$U_{em} = \frac{1}{2}\left[\int \left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right)\right] d\tau$$

Now suppose we have some charge and current configuration at time t, produces fields **E** and **B**. In the next instant dt, the charges move around a bit. How much work, dW, is done by the electromagnetic forces acting on these charges in the interval dt?

According to Lorentz force law:

$$dW = \vec{F}.d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}).\vec{v}dt = q\vec{E}.\vec{v}dt$$

Now $q = \rho d\tau$ and $\rho\vec{v} = \vec{J}$, so the rate at which work is done on all the charges in a volume V is:

$$\frac{dW}{dt} = \int (\vec{E}.\vec{J}) d\tau$$

So $\vec{E}.\vec{J}$ is the work done per unit time per unit volume, i.e. Power delivered per unit volume.

$$\vec{E}.\vec{J} = \vec{E}.\left[\frac{1}{\mu_o}(\vec{\nabla} \times \vec{B}) - \epsilon_o \frac{\partial \vec{E}}{\partial t}\right]$$

$$\vec{\nabla}.(\vec{E} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{E}) - \vec{E}.(\vec{\nabla} \times \vec{B})$$

Form Faraday's law: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\vec{E}.(\vec{\nabla} \times \vec{B}) = -\vec{B}.\frac{\partial \vec{B}}{\partial t} - \vec{\nabla}.(\vec{E} \times \vec{B}) = -\frac{1}{2}\frac{\partial B^2}{\partial t} - \vec{\nabla}.(\vec{E} \times \vec{B})$$

So

$$\vec{E}.\vec{J} = \vec{E}.\left[\frac{1}{\mu_o}(\vec{\nabla} \times \vec{B}) - \epsilon_o \frac{\partial \vec{E}}{\partial t}\right] = -\frac{1}{2\mu_o}\frac{\partial B^2}{\partial t} - \frac{1}{\mu_o}\vec{\nabla}.(\vec{E} \times \vec{B}) - \frac{\epsilon_o}{2}\frac{\partial E^2}{\partial t}$$

$$\vec{E}.\vec{J} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right) - \frac{1}{\mu_o}\vec{\nabla}.(\vec{E} \times \vec{B})$$

$$\frac{dW}{dt} = \int (\vec{E}.\vec{J}) d\tau = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right) d\tau - \frac{1}{\mu_o}\oint_S (\vec{E} \times \vec{B}).d\vec{a}$$

Where S is the surface bounding the volume V. And this is the **Poynting Theorem**, it is the work-energy theorem of electrodynamics. The first integral is the total energy stored in the electromagnetic field and the second term represents the rate at which energy is carried out of the volume V, across its boundary surface, by the electromagnetic field.

The **Poynting Theorem** states that **"The work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy flowed out through the surface"**

The energy per unit time, per unit area, transported by the fields is called **Poynting Vector**.

$$\vec{S} = \frac{1}{\mu_o}(\vec{E} \times \vec{B})$$

And $\vec{S}.d\vec{a}$ is the energy per unit time across the infinitesimal surface da- the energy flux, so $\vec{S}$ is the **energy flux density**.

We can write Poynting's Theorem more compactly as:

$$\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S}.d\vec{a}$$

Since the work done on the charges will increase the mechanical energy (kinetic, potential etc), if u_{mech} is the mechanical energy density, so

$$\frac{dW}{dt} = \frac{d}{dt}\int_V u_{mech}d\tau$$

And energy density of the electromagnetic fields is:

$$u_{em} = \frac{1}{2}\left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right)$$

$$\frac{d}{dt}\int_V (u_{mech} + u_{em})d\tau = -\oint_S \vec{S}.d\vec{a} = -\int_V (\vec{\nabla}.\vec{S})d\tau$$

$$\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -(\vec{\nabla}.\vec{S})$$

This is differential form of Poynting's theorem. Comparing it with the continuity equation:

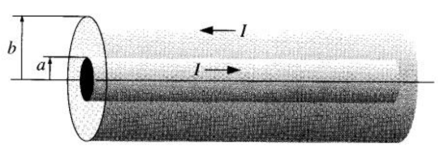
$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}.\vec{J}$$

The charge density is replaced by the energy density (mechanical+electromagnetic) and the current density is replaced by the Poynting vector. This means that flow of energy through any volume is the same as flow of charge through any volume.

Example 1:

Calculate the power (energy per unit time) transported down a long coaxial cable carrying current I (the current flows down the surface of the inner cylinder of radius *a* and back along the outer cylinder of radius *b*), assuming the two cylinders are kept at potential difference V.

Solution:



Electric field between the cylinders:

$$\vec{E} = \frac{1}{2\pi\epsilon_o}\frac{\lambda}{s}\hat{s}$$

$$\vec{B} = \frac{\mu_o I}{2\pi s}\hat{\phi}$$

So

$$\vec{S} = \frac{1}{\mu_o}(\vec{E} \times \vec{B}) = \left(\frac{\lambda I}{4\pi^2\epsilon_o}\frac{1}{s^2}\hat{k}\right)$$

$$Power = P = \oint \vec{S}.d\vec{a} = \int_a^b S * 2\pi s * ds = \int_a^b \frac{\lambda I}{2\pi\epsilon_o}\frac{1}{s} * ds = \frac{\lambda I}{2\pi\epsilon_o}\ln\left(\frac{b}{a}\right)$$

$$V = \int_a^b \vec{E}.d\vec{l} = \frac{\lambda}{2\pi\epsilon_o}\int_a^b \frac{1}{s}ds = \frac{\lambda}{2\pi\epsilon_o}\ln\left(\frac{b}{a}\right)$$

$$P = VI = \frac{\lambda I}{2\pi\epsilon_o}\ln\left(\frac{b}{a}\right)$$

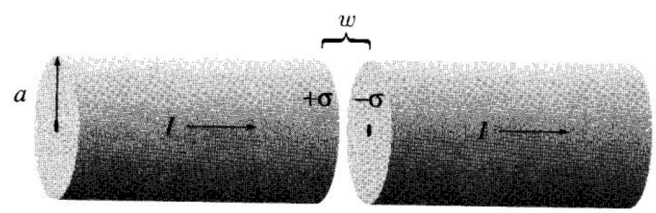
Example 2:

A wire of radius *a* carries a constant current I, uniformly distributed over its cross-section. A narrow gap in the wire of width w<<*a* , forms a parallel plate capacitor as shown in the figure.

(a) Find the electric and magnetic field in the gap as a function of the distance s from the axis and the time t.

(b) Find the energy density u_{em} and the Poynting vector $\vec{\boldsymbol{S}}$ in the gap.

(c) Determine the total energy in the gap as a function of time. Calculate the total power flowing into the gap, for a volume of radius *b*<*a*.



Solution:

(a) Electric field in the gap is:

$$\vec{E}(t) = \frac{\sigma}{\epsilon_o}\hat{k} = \frac{Q(t)}{\pi a^2\epsilon_o}\hat{k} = \frac{It}{\pi a^2\epsilon_o}\hat{k}$$

$$\vec{\nabla} \times \vec{B} = \mu_o\left(\vec{J} + \epsilon_o\frac{\partial\vec{E}}{\partial t}\right)$$

$$\int \vec{B}.d\vec{l} = \int(\vec{\nabla} \times \vec{B}).d\vec{a} = \int \mu_o\left(\vec{J} + \epsilon_o\frac{\partial\vec{E}}{\partial t}\right).d\vec{a} = \mu_o\epsilon_o\frac{\partial E}{\partial t} * \pi s^2 = \frac{\mu_o I s^2}{a^2}$$

$$B * 2\pi s = \frac{\mu_o I s^2}{a^2} \rightarrow \vec{B} = \frac{\mu_o I s}{2\pi a^2}\hat{\phi}$$

(b)

$$u_{em} = \frac{1}{2}\left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right) = \frac{1}{2}\left[\epsilon_o\left(\frac{It}{\pi a^2\epsilon_o}\right)^2 + \frac{1}{\mu_o}\left(\frac{\mu_o I s}{2\pi a^2}\right)^2\right]$$

$$u_{em} = \frac{\mu_o I^2}{2\pi^2 a^4}\left[(ct)^2 + \left(\frac{s}{2}\right)^2\right]$$

$$\vec{S} = \frac{1}{\mu_o}\left(\vec{E} \times \vec{B}\right) = \frac{1}{\mu_o}\left(\frac{It}{\pi a^2 \epsilon_o}\right)\left(\frac{\mu_o I s}{2\pi a^2}\right)\left(\hat{k} \times \hat{\phi}\right) = -\frac{I^2 t * s}{2\pi^2 \epsilon_o a^4}\hat{s}$$

(c)

$$U_{em} = \frac{1}{2}\left[\int \left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right)\right] d\tau = \int u_{em} d\tau = \int u_{em} w * 2\pi s * ds$$

$$U_{em} = \int_0^b \frac{\mu_o I^2}{2\pi^2 a^4}\left[(ct)^2 + \left(\frac{s}{2}\right)^2\right] w * 2\pi s * ds = \frac{\mu_o I^2 w}{\pi a^4}\left[\frac{(ct)^2 s^2}{2} + \frac{1}{4}\frac{s^4}{4}\right]_0^b$$

$$U_{em} = \frac{\mu_o I^2 w b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right]$$

$$P_{in} = -\int \vec{S}.d\vec{a} = \frac{I^2 t}{2\pi^2 \epsilon_o a^4}(b\hat{s}.2\pi b w \hat{s}) = \frac{I^2 w t b^2}{\pi \epsilon_o a^4}$$

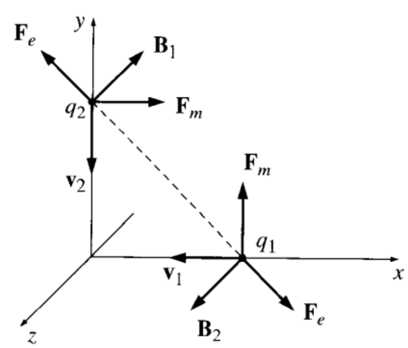
8.2 Momentum

8.2.1 Newton's Third Law in Electrodynamics

Imagine a point charge q travelling in along the x-axis at a constant speed v, since it is moving its electric field is not given by Coulomb's law but $\vec{E}$ still points radially outward from the instantaneous position of the charge.

Also a moving point charge does not constitute a steady current, its magnetic field is not given by Biot-Savart law but in fact $\vec{B}$ still circles around the axis in a manner suggested by the right hand rule.

Now assume there is another identical charge moving along the y-axis and they both interact with each other as shown in the figure. Let's assume they both cannot be driven out of their path of travelling due to electromagnetic forces on each other.



Now if we consider the electric force between them they will be same in magnitude but opposite in direction along the line joining the two charges.

But the magnetic force on q_2 due to q_1 will be to the right along x-axis and the magnetic force on q_1 due to q_2 will be upward along the y-axis. The magnitude to the magnetic forces is equal in magnitude but not opposite in direction, so they don't cancel each other.

Newton's third law fails here: means there is no conservation of momentum here.

Well, in electrodynamics the electromagnetic field themselves carry momentum as we have already calculated that electromagnetic fields carry energy. So if we take momentum of electromagnetic field and momentum of the charges then total momentum will be conserved. Hence, Newton's third law is not violated in Electrodynamics.

8.2.2 Maxwell's Stress Tensor

Let's calculate the total electromagnetic force on the charges in a volume V:

$$\vec{F} = \int \left(\vec{E} + \vec{v} \times \vec{B}\right)\rho d\tau = \int \left(\rho\vec{E} + \vec{J} \times \vec{B}\right)d\tau$$

The force per unit volume will be

$$\vec{f} = \rho\vec{E} + \vec{J} \times \vec{B}$$

The above expression can be written interms of fields by using Maxwell's equations:

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_o} \quad \text{and} \quad \vec{\nabla} \times \vec{B} = \mu_o\vec{J} + \mu_o\epsilon_o\frac{\partial\vec{E}}{\partial t}$$

$$\vec{f} = \epsilon_o\left(\vec{\nabla}.\vec{E}\right)\vec{E} + \left(\frac{1}{\mu_o}\vec{\nabla} \times \vec{B} - \epsilon_o\frac{\partial\vec{E}}{\partial t}\right) \times \vec{B}$$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right) + \left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)$$

Faraday's law says:

$$\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

$$\frac{\partial\vec{E}}{\partial t}\times\vec{B} = \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times(\vec{\nabla}\times\vec{E})$$

$$\vec{f} = \epsilon_o[(\vec{\nabla}.\vec{E})\vec{E} - \vec{E}\times(\vec{\nabla}\times\vec{E})] - \frac{1}{\mu_o}\left(\vec{\mathrm{B}}\times(\vec{\nabla}\times\vec{B})\right) - \epsilon_o\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\vec{\nabla}(E^2) = 2\left(\vec{\mathrm{E}}.\vec{\nabla}\right)\vec{E} + 2\vec{\mathrm{E}}\times\left(\vec{\nabla}\times\vec{\mathrm{E}}\right)$$

$$\vec{\mathrm{E}}\times\left(\vec{\nabla}\times\vec{\mathrm{E}}\right) = \frac{1}{2}\vec{\nabla}(E^2) - \left(\vec{\mathrm{E}}.\vec{\nabla}\right)\vec{E}$$

Similarly,

$$\vec{\mathrm{B}}\times\left(\vec{\nabla}\times\vec{B}\right) = \frac{1}{2}\vec{\nabla}(B^2) - \left(\vec{\mathrm{B}}.\vec{\nabla}\right)\vec{B}$$

$$\vec{f} = \epsilon_o\left[(\vec{\nabla}.\vec{E})\vec{E} - \frac{1}{2}\vec{\nabla}(E^2) + \left(\vec{\mathrm{E}}.\vec{\nabla}\right)\vec{E}\right] - \frac{1}{\mu_o}\left(\frac{1}{2}\vec{\nabla}(B^2) - \left(\vec{\mathrm{B}}.\vec{\nabla}\right)\vec{B}\right) - \epsilon_o\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\vec{f} = \epsilon_o\left((\vec{\nabla}.\vec{E})\vec{E} + \left(\vec{\mathrm{E}}.\vec{\nabla}\right)\vec{E}\right) + \frac{1}{\mu_o}\left((\vec{\nabla}.\vec{B})\vec{B} + \left(\vec{\mathrm{B}}.\vec{\nabla}\right)\vec{B}\right) - \frac{1}{2}\vec{\nabla}\left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right) - \epsilon_o\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

$$\vec{\nabla}.\vec{B} = 0$$

If we use Maxwell's Stress Tensor:

$$T_{ij} = \epsilon_o\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_o}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

The indices i and j refer to the coordinates x, y, and z, so the stress tensor has a total of nine components $(T_{xx}, T_{yy}, T_{zz}, T_{xy}, T_{xz}, \ldots)$. The Kronecker delta, δ_{ij} is 1 if the indices are the same ($\delta_{xx} = \delta_{yy} = \delta_{zz} = 1$) and zero otherwise.

$$T_{xx} = \frac{1}{2}\epsilon_o\left(E_x^2 - E_y^2 - E_z^2\right) + \frac{1}{2\mu_o}\left(B_x^2 - B_y^2 - B_z^2\right)$$

$$T_{xy} = \epsilon_o\left(E_xE_y\right) + \frac{1}{\mu_o}\left(B_xB_y\right)$$

And so on. A vector carries one index and tensor carries two indices. A dot product between a tensor and a vector can be written as:

$$\left(\vec{a}.\overleftrightarrow{T}\right)_j = \sum_{i=x,y,z} a_iT_{ij}$$

The resulting object that has only one index is itself a vector. The divergence of tensor $\overleftrightarrow{T}$ has as its jth component as:

$$\left(\vec{\nabla}.\overleftrightarrow{T}\right)_j = \epsilon_o\left[(\vec{\nabla}.\vec{E})E_j + (\vec{\mathrm{E}}.\vec{\nabla})E_j - \frac{1}{2}\nabla_j(E^2)\right] + \frac{1}{\mu_o}\left[(\vec{\nabla}.\vec{B})B_j + (\vec{\mathrm{B}}.\vec{\nabla})B_j - \frac{1}{2}\nabla_j(B^2)\right]$$

$$\vec{f} = \vec{\nabla}.\overleftrightarrow{T} - \epsilon_o\mu_o\frac{\partial\vec{S}}{\partial t}$$

Where $\vec{S} = \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right)$ is the **Poynting vector**.

The total force on the charges is:

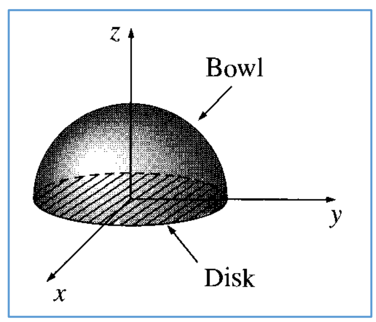
$$\vec{F} = \int_V (\vec{\nabla}.\overleftrightarrow{T})d\tau - \epsilon_o\mu_o\frac{d}{dt}\int_V \vec{S}\,d\tau = \oint_S \overleftrightarrow{T}.d\vec{a} - \epsilon_o\mu_o\frac{d}{dt}\int_V \vec{S}\,d\tau$$

In the static case or whenever $\int\vec{S}\,d\tau$ is independent of time the force on the charges will be entirely given by the stress tensor at the boundary. Physically $\overleftrightarrow{T}$ is the force per unit area (or stress) acting on the surface. And $\boldsymbol{T_{ij}}$ **is the force per unit area in the ith direction acting on an element of surface oriented in the jth direction**.

The diagonal elements (T_{xx}, T_{yy}, T_{zz}) represent **Pressures** and off-diagonal elements $(T_{xy}, T_{yz}, T_{zx}, \ldots)$ are **shears**.

Example 3:

Determine the net force on the northern hemisphere of a uniformly charged solid sphere of radius R and charge Q.



Solution:

Since the net force on the bowl is:

$$\vec{F} = \oint\overleftrightarrow{T}.d\vec{a} - \epsilon_o\mu_o\frac{d}{dt}\int\vec{S}\,d\tau$$

Since the force is only along z-axis, so we need to calculate $\left(\overleftrightarrow{T}.d\vec{a}\right)_z$

The boundary surface consists of two parts- a hemispherical “bowl” at radius R, and a circular disk at $\theta = \frac{\pi}{2}$.

For the bowl, $\quad da = R^2\sin\theta\;d\theta\,d\phi\,\hat{r}$

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{Q}{R^2}\hat{r}$$

In Cartesian components, $\quad \hat{r} = \sin\theta\cos\phi\;\hat{\imath} + \sin\theta\sin\phi\;\hat{\jmath} + \cos\theta\,\hat{k}$

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{Q}{R^2}\left[\sin\theta\cos\phi\;\hat{\imath} + \sin\theta\sin\phi\;\hat{\jmath} + \cos\theta\,\hat{k}\right]$$

$$T_{zx} = \epsilon_o(E_zE_x) + \frac{1}{\mu_o}(B_zB_x) = \epsilon_o\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R^2}\right)^2 \sin\theta\cos\phi\cos\theta$$

$$T_{zy} = \epsilon_o\left(E_zE_y\right) = \epsilon_o\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R^2}\right)^2 \sin\theta\sin\phi\cos\theta$$

$$T_{zz} = \frac{1}{2}\epsilon_o\left(E_z^2 - E_x^2 - E_y^2\right) + \frac{1}{2\mu_o}\left(B_z^2 - B_x^2 - B_y^2\right)$$

$$T_{zz} = \frac{\epsilon_o}{2}\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R^2}\right)^2 (\cos^2\theta - \sin^2\theta)$$

The net force is in the z-direction so

$$\left(\overleftrightarrow{T}.d\vec{a}\right)_z = T_{zx}da_x + T_{zy}da_y + T_{zz}da_z = \frac{\epsilon_o}{2}\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R^2}\right)^2 R^2\sin\theta\cos\theta\,d\theta\,d\phi$$

The force on the bowl is :

$$F_{bowl} = \oint\left(\overleftrightarrow{T}.d\vec{a}\right)_z - \epsilon_o\mu_o\frac{d}{dt}\int\vec{S}\,d\tau = \oint\left(\overleftrightarrow{T}.d\vec{a}\right)_z = \int_0^{2\pi}\int_0^{\pi/2}\frac{\epsilon_o}{2}\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R^2}\right)^2 R^2\sin\theta\cos\theta\,d\theta\,d\phi$$

$$F_{Bowl} = \frac{\epsilon_o}{2}\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R}\right)^2 * 2\pi * \int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,d\theta \;= \frac{1}{4\pi\epsilon_o}\frac{Q^2}{8R^2}$$

For the equatorial disk,

$$d\vec{a} = -r\,dr\,d\phi\,\hat{k}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{Q}{R^3}r\hat{r} = \frac{1}{4\pi\epsilon_o}\frac{Qr}{R^3}\left(\sin\theta\cos\phi\;\hat{\imath} + \sin\theta\sin\phi\;\hat{\jmath} + \cos\theta\,\hat{k}\right)$$

For $\theta = \frac{\pi}{2}$; $\vec{E} = \frac{1}{4\pi\epsilon_o}\frac{Qr}{R^3}(\cos\phi\;\hat{\imath} + \sin\phi\;\hat{\jmath})$

$$T_{zz} = \frac{\epsilon_o}{2}\left(E_z^2 - E_x^2 - E_y^2\right) = -\frac{\epsilon_o}{2}\left(\frac{1}{4\pi\epsilon_o}\frac{Qr}{R^3}\right)^2(\cos^2\phi + \sin^2\phi) = -\frac{\epsilon_o}{2}\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R^3}\right)^2 r^2$$

$$\left(\overleftrightarrow{T}.d\vec{a}\right)_z = T_{zx}da_x + T_{zy}da_y + T_{zz}da_z = \frac{\epsilon_o}{2}\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R^3}\right)^2 r^3\,dr\,d\phi$$

$$F_{disk} = \frac{\epsilon_o}{2}\left(\frac{1}{4\pi\epsilon_o}\frac{Q}{R^3}\right)^2\int_0^{2\pi}\int_0^R r^3\,dr\,d\phi = \frac{1}{4\pi\epsilon_o}\frac{Q^2}{16R^2}$$

The net force on the norther hemisphere would be:

$$F = F_{bowl} + F_{disk} = \frac{1}{4\pi\epsilon_o}\frac{Q^2}{8R^2} + \frac{1}{4\pi\epsilon_o}\frac{Q^2}{16R^2} = \frac{1}{4\pi\epsilon_o}\frac{3Q^2}{16R^2}$$

8.2.3 Conservation of Momentum

According to Newton's second law, the force on an object is equal to the rate of change of its momentum:

$$\vec{F} = \frac{d\vec{p}_{mech}}{dt} = -\epsilon_o\mu_o \frac{d}{dt}\int_{S} \vec{S}\, d\tau + \oint_{V} \overleftrightarrow{T}.d\vec{a}$$

Where $\vec{p}_{mech}$ is the total (mechanical) momentum of the particles contained in volume V. This expression is similar in structure to Poynting's theorem.

The first integral represents momentum stored in the electromagnetic fields themselves and the second integral is the momentum per unit time flowing in through the surface.

$$\vec{p}_{em} = \epsilon_o\mu_o \int \vec{S}\, d\tau$$

Any increase in the total momentum (mechanical plus electromagnetic) is equal to the momentum brought in by the fields. If V is all of space, then no momentum flows in or out, and $\vec{p}_{mech} + \vec{p}_{em}$ is constant.

Let $\vec{\wp}_{mech}$ be the density of mechanical momentum and $\vec{\wp}_{em}$ the density of momentum in the fields:

$$\vec{\wp}_{em} = \mu_o\epsilon_o\vec{S}$$

The equation in differential forms says:

$$\frac{\partial}{\partial t}\left(\vec{\wp}_{mech} + \vec{\wp}_{em}\right) = \vec{\nabla}.\overleftrightarrow{T}$$

Evidently $-\overleftrightarrow{T}$ is the momentum flux density, playing the role of $\vec{J}$ in the continuity equation or $\vec{S}$ in Poynting's theorem.

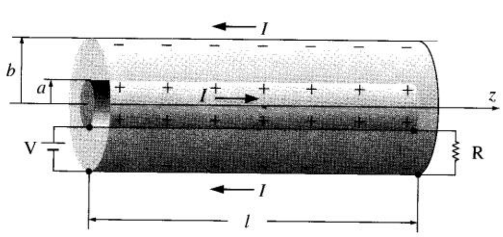
Specifically $-T_{ij}$ is the momentum in the *i* direction crossing a surface oriented in the *j* direction, per unit area per unit time.

$\vec{S}$ itself is the energy per unit area per unit time transported by the electromagnetic fields, while $\mu_o\epsilon_o\vec{S}$ is the momentum per unit volume stored in the fields.

Similarly, $\overleftrightarrow{T}$ itself is the electromagnetic stress (force per unit area) acting on the surface and $-\overleftrightarrow{T}$ describes the flow of momentum (the momentum current density) transported by the fields.

Example 4:

A long coaxial cable of length *l* consists of an inner conductor of radius *a* and an outer conductor of radius b. It is connected to a battery at one end and a resistor at the other end. The inner conductor carries a uniform charge per unit length λ and a steady current I to the right; the outer conductor has the opposite charge and current. What is the electromagnetic momentum stored in the fields?



Solution:

The fields are:

$$\vec{E} = \frac{1}{2\pi\epsilon_o}\frac{\lambda}{s}\hat{s} \quad \text{and} \quad \vec{B} = \frac{\mu_o}{2\pi}\frac{I}{s}\hat{\phi}$$

The poynting vector is:

$$\vec{S} = \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right) = \frac{\lambda I}{4\pi^2\epsilon_o s^2}\hat{k}$$

Evidently energy is flowing down the line from the battery to the resistor. In fact, the power transported is:

$$P = \int \vec{S}.d\vec{a} = \frac{\lambda I}{4\pi^2\epsilon_o s^2}\int_a^b \frac{2\pi s\, ds}{s^2} = \frac{\lambda I}{2\pi\epsilon_o}\ln\left(\frac{b}{a}\right) = IV$$

The momentum in the fields is:

$$\vec{p}_{em} = \epsilon_o\mu_o\int \vec{S}\,d\tau = \epsilon_o\mu_o\frac{\lambda I}{4\pi^2\epsilon_o s^2}\hat{k}\int_a^b \frac{l2\pi s\, ds}{s^2} = \frac{\mu_o\lambda I l}{2\pi}\ln\left(\frac{b}{a}\right)\hat{k}$$

This is the momentum in the field, although the cable is not moving but there is a hidden momentum due to flow of charges in the cable which cancels the momentum in the field.

Suppose now we decrease the current to zero by increasing the resistance, for example, then change in magnetic field will induce an electric field, which is given as:

$$\oint \vec{E}.d\vec{l} = -\int \frac{\partial\vec{B}}{\partial t}.d\vec{a} = -\int \frac{\mu_o}{2\pi s}\frac{dI}{dt}\,da$$

$$E(a)l - E(s)l = -\frac{\mu_o}{2\pi}\frac{dI}{dt}\int_a^s \frac{l}{s}\,ds = -\frac{\mu_o l}{2\pi}\frac{dI}{dt}\left[\ln(s) - \ln(a)\right]$$

$$\vec{E}(s) = \left[\frac{\mu_o}{2\pi}\frac{dI}{dt}\ln(s) + K\right]\hat{k}$$

This electric field will exert force on $\pm\lambda$:

$$\vec{F} = q\vec{E} = \lambda l\left[\frac{\mu_o}{2\pi}\frac{dI}{dt}\ln(a) + K\right]\hat{k} - \lambda l\left[\frac{\mu_o}{2\pi}\frac{dI}{dt}\ln(b) + K\right]\hat{k} = \frac{\mu_o\lambda l}{2\pi}\frac{dI}{dt}\ln\left(\frac{b}{a}\right)\hat{k}$$

The total momentum imparted to the cable as the current drops from I to 0, is:

$$\vec{p}_{mech} = \int \vec{F}\,dt = \frac{\mu_o\lambda l I}{2\pi}\ln\left(\frac{b}{a}\right)\hat{k}$$

Which is exactly the same momentum originally stored in the field. But the cable will not recoil because there is equal and opposite impulse delivered by the simultaneous disappearance of the hidden momentum.

Example 5:

Consider an infinite parallel-plate capacitor, with the lower plate at $z = -d/2$ carrying the charge density $-\sigma$ and the upper plate at $z = +d/2$ carrying the charge density $+\sigma$.

a) Determine all nine elements of the stress tensor, in the region between the plates, in 3x3 matrix form.
b) Find the force per unit area on the top plate.
c) What is the momentum per unit area per unit time crossing the xy-plane (or any other plane parallel to that one, between the plates).
d) At the plates, this momentum is absorbed, and the plates recoil (unless there is some nonelectrical force holding them in position). Find the recoil force per unit area on the top plate and compare your answer to part b.

Solution:

a) $E_x = E_y = 0$ and $E_z = -\frac{\sigma}{\epsilon_o}$

Since $$T_{ij} = \epsilon_o\left(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \tfrac{1}{\mu_o}\left(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2\right)$$

Therefore $T_{xy} = T_{xz} = T_{zy} = \cdots = 0$ whereas $T_{xx} = T_{yy} = -\frac{\epsilon_o}{2}E^2 = -\frac{\sigma^2}{2\epsilon_o}$ and

$$T_{zz} = \epsilon_o\left(E_z^2 - \frac{1}{2}E^2\right) = \frac{\epsilon_o}{2}E^2 = \frac{\sigma^2}{2\epsilon_o}$$

$$\overleftrightarrow{T} = \frac{\sigma^2}{2\epsilon_o}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

b) $$\vec{F} = -\epsilon_o\mu_o\frac{d}{dt}\int \vec{S}\,d\tau + \oint \overleftrightarrow{T}.d\vec{a}$$

Since there is no magnetic field, hence $\vec{S} = \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right) = 0$

So, $$\vec{F} = \oint \overleftrightarrow{T}.d\vec{a} \quad \text{where} \quad d\vec{a} = -dxdy\hat{k}$$

$$F_z = \int T_{zz}da_z = -\frac{\sigma^2}{2\epsilon_o}A$$

And force per unit area would be: $f = \frac{F_z}{A} = -\frac{\sigma^2}{2\epsilon_o}\hat{k}$

c) Momentum in the z-direction per unit time per unit area would be: $-T_{zz} = -\frac{\sigma^2}{2\epsilon_o}$

d) The recoil force is the momentum delivered per unit time, so the force per unit area on the top plate would be: $f = -\frac{\sigma^2}{2\epsilon_o}\hat{k}$

This is same as we got in part (b)

8.2.4 Angular Momentum

Since the electromagnetic fields carry energy and momentum as given by:

$$u_{em} = \frac{1}{2}\left(\epsilon_o E^2 + \frac{1}{\mu_o} B^2\right)$$

$$\vec{\wp}_{em} = \mu_o \epsilon_o \vec{S} = \epsilon_o (\vec{E} \times \vec{B})$$

So, we can calculate angular momentum as well:

$$\vec{l}_{em} = \vec{r} \times \vec{\wp}_{em} = \epsilon_o [\vec{r} \times (\vec{E} \times \vec{B})]$$

So even perfectly static fields can carry momentum and angular momentum, as long as $\vec{E} \times \vec{B} \neq 0$, and the classical conservation laws are upheld when the contribution from these static fields are included.

Example 5:

Imagine a very long solenoid of radius R with n turns per unit length and current I. Coaxial with the solenoid are two long cylindrical shells of length l - one, inside the solenoid at radius *a*, carries a charge +Q, uniformly distributed over its surface, the other outside the solenoid at radius *b*, that carries a charge –Q. When the current in the solenoid is gradually reduced, the cylinders begin to rotate. Where does the angular momentum come from?

Solution:

$$\vec{E} = \frac{Q}{2\pi\epsilon_o l}\frac{1}{s}\hat{s} \;\; (a < s < b)$$

In the region between the cylinders (inside the solenoid):

$$\vec{B} = \mu_o n I \, \hat{k} \;\; (s < R)$$

The momentum density is therefore:

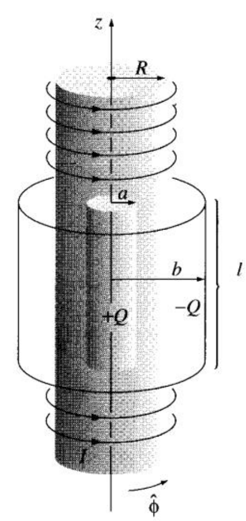
$$\vec{\wp}_{em} = \mu_o \epsilon_o \vec{S} = \epsilon_o (\vec{E} \times \vec{B}) = -\frac{\mu_o n I Q}{2\pi l s} \, \hat{\phi}$$

In the region $a < s < R$, the angular momentum density is:

$$\vec{l}_{em} = \vec{s} \times \vec{\wp}_{em} = -\frac{\mu_o n I Q}{2\pi l}\hat{k}$$

To get the total angular momentum in the fields, we simply multiply by the volume $\pi(R^2 - a^2)l$:

$$\vec{L}_{em} = -\frac{1}{2}\mu_o n I Q (R^2 - a^2)\hat{k}$$



When the current is turned off, the changing magnetic field induces a circumferential electric field, given by Faraday's law:

$$\oint \vec{E}.d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}.d\vec{a} = -\mu_o n \frac{dI}{dt} * \pi s^2$$

$$E * \cancel{2\pi s} = -\mu_o n \frac{dI}{dt} * \frac{\cancel{\pi} s^2}{2s}\ \hat{\phi}$$

$$\vec{E} = \begin{cases} -\frac{1}{2}\mu_o n \frac{dI}{dt} * \frac{R^2}{s}\ \hat{\phi} & (s > R) \\ -\frac{1}{2}\mu_o n \frac{dI}{dt} s\ \hat{\phi} & (s < R) \end{cases}$$

Thus the torque on the outer cylinder is:

$$\vec{N}_b = \vec{r} \times \vec{F} = \vec{r} \times \left(-Q\vec{E}\right) = \frac{1}{2}\mu_o n Q R^2 \frac{dI}{dt}\hat{k}$$

And it picks up the angular momentum,

$$\vec{L}_b = \frac{1}{2}\mu_o n Q R^2 \hat{k} \int_I^0 \frac{dI}{dt} dt = -\frac{1}{2}\mu_o n Q I R^2 \hat{k}$$

Similarly the torque on the inner cylinder is:

$$\vec{N}_a = \vec{r} \times \vec{F} = \vec{r} \times \left(+Q\vec{E}\right) = -\frac{1}{2}\mu_o n Q a^2 \frac{dI}{dt}\hat{k}$$

And its angular momentum increase is:

$$\vec{L}_a = -\frac{1}{2}\mu_o n Q a^2 \hat{k} \int_I^0 \frac{dI}{dt} dt = \frac{1}{2}\mu_o n Q I a^2 \hat{k}$$

$$\vec{L}_a + \vec{L}_b = -\frac{1}{2}\mu_o n Q I (R^2 - a^2)\hat{k} = \vec{L}_{em}$$

The angular momentum lost by the field is precisely equal to the angular momentum gained by the cylinders, and hence the total angular momentum is conserved.