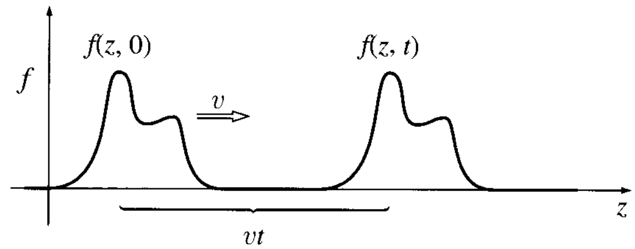
Chapter 9 - Electromagnetic Waves

9.1 Waves in One Dimension

Figure below shows a wave drawn at two different times, one at t=0 and the other at a later time t.

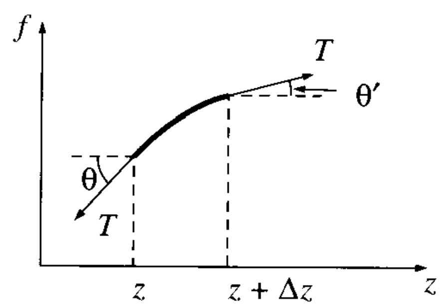


The shape is same at two different times. The displacement at point z, at the later time t is the same as the displacement a distance vt to the left $(z - vt)$ back at time $t = 0$.

$$f(z,t) = f(z - vt, 0) = g(z - vt)$$

Where $g(z - vt)$ is the initial shape of the string at t=0. This means that the function $f(z,t)$ which might depend on z and t in any old way, in fact depends on them in the very special combination $z - vt$; when that is true, the function $f(z,t)$ represents a wave of a fixed shape travelling in the z-direction at speed v.

When a taut string is displaced from its equilibrium position, a transverse force on the segment between z and z+Δz is:



$$\Delta F = T\sin\theta' - T\sin\theta$$

Where θ' is the angle the string makes with the z-direction at $z + \Delta z$ and θ is the angle that string makes with the z-axis at point z. If the distortion is not too great then these angles are small and we can replace the sine functions with tangent functions:

$$\Delta F = T\tan\theta' - T\tan\theta = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z$$

If the mass per unit length is μ, Newton's second law says:

$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}$$

$$T\frac{\partial^2 f}{\partial z^2}\Delta z = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2}\ ; \qquad \frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Where v is the speed of the wave on the string. The above equation is called **Wave equation** because it admits as solutions all functions of the form:

$$f(z,t) = g(z - vt)$$

Since the wave equation involves the square of v, so another set of solutions can be generated by changing the sign of v:

$$f(z,t) = h(z + vt)$$

Which represents the waves propagating in the negative z-direction. The general wave solution, however, would be:

$$f(z,t) = g(z - vt) + h(z + vt)$$

Example 1:

Show that the standing wave $f(z,t) = A\sin(kz)\cos(kvt)$ satisfies the wave equation and express it as the sum of a wave travelling to the left and a wave travelling to the right.

9.1.2 Sinusoidal Waves

A sinusoidal wave can be written in the following form:

$$f(z,t) = A\cos(kz - \omega t + \delta)$$

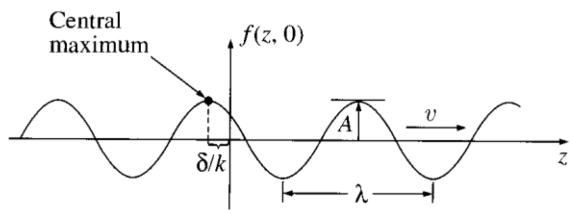
Where k is the wavenumber which is related to the wavelength as:

$$\lambda = \frac{2\pi}{k}$$

And ω is the angular frequency given as:

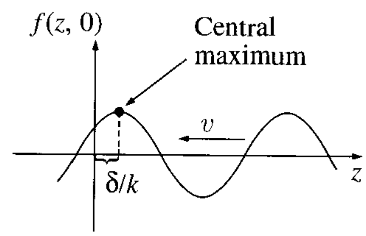
$$\omega = 2\pi\nu = kv$$

And δ is the phase constant where $\frac{\delta}{k}$ is the distance for which wave is delayed.



In the figure shown $= vt - \delta/k$, the phase is zero; lets call this the "central maximum".

If $\delta = 0$, the central maximum passes through the origin at time t=0, so δ/k is the distance by which the central maximum and the whole wave is delayed.



In view of the Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The sinusoidal wave can be written as:

$$f(z,t) = Re\left[Ae^{i(kz-\omega t+\delta)}\right]$$

The complex wavefunction can be written as:

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$$

Where $\tilde{A} = Ae^{i\delta}$ is the complex amplitude. Hence

$$f(z,t) = Re\left[\tilde{f}(z,t)\right]$$

Any wave can be expressed as a linear combination of sinusoidal ones:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}\,(k)e^{i(kz-\omega t)}dk$$

9.1.3 Boundary Conditions: Reflections and Transmission

When the string is not infinitely long then when the wave propagates and reaches at the end of the string then it matters how the string is attached at the end. For example the string is simply tied to another second string. The tension is the same for both the strings but the linear mass density is different so speed of the wave will be different in the two string. Let's say the knot is at z=0, then the **incident wave** will be:

$$\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)} \quad \text{for} \quad (z < 0)$$

Coming in from the left, gives rise to a **reflected wave** along the same string 1,

$$\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)} \quad \text{for} \quad (z < 0)$$

And part of the incident wave will be **transmitted**, which continues to the right in string 2.

$$\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)} \quad \text{for} \quad (z > 0)$$

$f_I(z,t)$ is a sinusoidal oscillation that extends to $z = -\infty$, similarly reflected wave will go back to $z = -\infty$ and the transmitted wave will go to $z = +\infty$. All parts of the system are oscillating at the same frequency.

For a sinusoidal incident wave, the net disturbance of the string is:

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & for\ z < 0 \\ \tilde{A}_T e^{i(k_2 z - \omega t)}, & for\ z > 0 \end{cases}$$

At the joint, $(z = 0)$ the displacement just left and just right of $z = 0$ must be same, else there would be a break between the two strings:

$$f(0^-,t) = f(0^+,t)$$

If the knot itself has negligible mass then the derivative of f must also be continuous, else there would be net force on the knot:

$$\left.\frac{\partial f}{\partial z}\right|_{0^-} = \left.\frac{\partial f}{\partial z}\right|_{0^+}$$

These boundary conditions apply to the real wave functions and apply to complex wave functions as well:

$$\tilde{f}(0^-,t) = \tilde{f}(0^+,t) \ ; \ \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+}$$

Using the boundary condition on the wave equation, we can get the outgoing amplitudes ($\tilde{A}_R\ and\ \tilde{A}_T$) in terms of the incoming amplitude ($\tilde{A}_I$):

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T$$

$$k_1(\tilde{A}_I - \tilde{A}_R) = k_2\tilde{A}_T$$

Or

$$\tilde{A}_R = \left(\frac{k_1-k_2}{k_1+k_2}\right)\tilde{A}_I \quad \text{and} \quad \tilde{A}_T = \left(\frac{2k_1}{k_1+k_2}\right)\tilde{A}_I$$

Or interms of velocities:

$$\tilde{A}_R = \left(\frac{v_2-v_1}{v_1+v_2}\right)\tilde{A}_I \quad \text{and} \quad \tilde{A}_T = \left(\frac{2v_2}{v_1+v_2}\right)\tilde{A}_I$$

The real amplitudes and phases are then related as:

$$A_R e^{i\delta_R} = \left(\frac{v_2-v_1}{v_1+v_2}\right)A_I e^{i\delta_I} \quad \text{and} \quad A_T e^{i\delta_T} = \left(\frac{2v_2}{v_1+v_2}\right)A_I e^{i\delta_I}$$

If the second string is lighter than the first ($\mu_2 < \mu_1$, so that $v_2 > v_1$), all three waves have the same phase angle ($\delta_R = \delta_T = \delta_I$), and the outgoing amplitudes are:

$$A_R = \left(\frac{v_2-v_1}{v_1+v_2}\right)A_I \quad \text{and} \quad A_T = \left(\frac{2v_2}{v_1+v_2}\right)A_I$$

If the second string is heavier than the firs ($v_2 < v_1$) the reflected wave is out of phase by 180° ($\delta_R + \pi = \delta_T = \delta_I$)

$$\cos(-k_1 z - \omega t + \delta_I + \pi) = -\cos(-k_1 z - \omega t + \delta_I)$$

It means that reflected wave is upside down, the amplitudes in this case are:

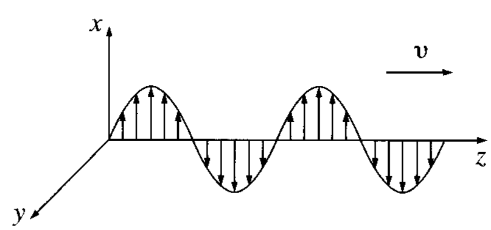
$$A_R = \left(\frac{v_1-v_2}{v_1+v_2}\right)A_I \quad \text{and} \quad A_T = \left(\frac{2v_2}{v_1+v_2}\right)A_I$$

If the second string is infinitely massive, if the first string is nailed down at the end, then

$$A_R = A_I \quad \text{and} \quad A_T = 0$$

9.1.4 Polarization

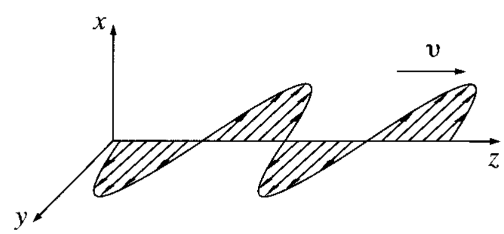
The waves in which displacement is perpendicular to the direction of the wave travel are called transverse waves and the waves in which displacement are parallel to the wave direction are called longitudinal wave.



The transverse waves occur in two independent states of polarization, we can oscillate the string up-and-down so it will be a vertical polarization.

$$\tilde{f}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\imath}$$

Or we can oscillate the string horizontally, so it will be horizontal polarization:



$$\tilde{f}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{\jmath}$$

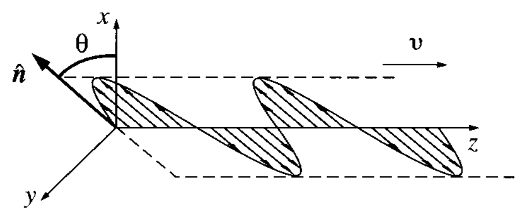
Or along any direction in the xy-plane:

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}$$

The polarization vector $\hat{n}$ defines the plane of polarization, because the waves are transverse, $\hat{n}$ is perpendicular to the direction of the wave travel:

$$\hat{n}.\hat{k} = 0$$

In terms of polarization angle θ,



$$\hat{n} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$$

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}$$

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

$$\tilde{f}(z,t) = \left(\tilde{A}\cos\theta\right)e^{i(kz-\omega t)}\hat{\imath} + (\tilde{A}\sin\theta)e^{i(kz-\omega t)}\hat{\jmath}$$

9.2 Electromagnetic Waves in Vacuum

In a region of space there is no charge or current, hence Maxwell's equation would become:

$$\vec{\nabla}.\vec{\mathrm{E}} = 0 \quad ; \quad \vec{\nabla}.\vec{\mathrm{B}} = 0$$

$$\vec{\nabla} \times \vec{\mathrm{E}} = -\frac{\partial \vec{B}}{\partial t} \quad ; \quad \vec{\nabla} \times \vec{\mathrm{B}} = \mu_o \epsilon_o \frac{\partial \vec{E}}{\partial t}$$

These equations are set of coupled partial differential equations for $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$. They can be decoupled using following identities:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathrm{E}}) = \vec{\nabla}(\vec{\nabla}.\vec{\mathrm{E}}) - \nabla^2\vec{\mathrm{E}} = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\mathrm{B}}) = -\mu_o\epsilon_o\frac{\partial^2 E}{\partial t^2}$$

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathrm{E}}) - \nabla^2\vec{\mathrm{E}} = -\mu_o\epsilon_o\frac{\partial^2 E}{\partial t^2}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathrm{B}}) = \vec{\nabla}(\vec{\nabla}.\vec{\mathrm{B}}) - \nabla^2\vec{\mathrm{B}} = \vec{\nabla} \times \left(\mu_o\epsilon_o\frac{\partial \vec{E}}{\partial t}\right) = \mu_o\epsilon_o\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{\mathrm{E}}) = -\mu_o\epsilon_o\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$\vec{\nabla}(\vec{\nabla}.\vec{\mathrm{B}}) - \nabla^2\vec{\mathrm{B}} = -\mu_o\epsilon_o\frac{\partial^2 \vec{B}}{\partial t^2}$$

Since $\vec{\nabla}.\vec{\mathrm{E}} = 0$ and $\vec{\nabla}.\vec{\mathrm{B}} = 0$, so

$$\nabla^2\vec{\mathrm{E}} = \mu_o\epsilon_o\frac{\partial^2 E}{\partial t^2} \quad \text{and} \quad \nabla^2\vec{\mathrm{B}} = \mu_o\epsilon_o\frac{\partial^2 \vec{B}}{\partial t^2}$$

These are separate equation for $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ but now the equations are second order differential equations.

In vacuum, each Cartesian component of $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ satisfy the three-dimensional wave equation:

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Therefore, the above Maxwell's equation would imply that:

$$v = \frac{1}{\sqrt{\mu_o \epsilon_o}} = 3.00 \times 10^8\ m/s$$

Which is exactly the speed of light c, which implies that light is an electromagnetic wave.

9.2.2 Monochromatic Plane Waves

The sinusoidal waves that are travelling in the z-direction and have no dependence on x and y direction are called plane waves because the fields are uniform over any plane perpendicular to the direction of propagation.

$$\tilde{E}(z,t) = \tilde{E}_o e^{i(kz-\omega t)} \qquad ; \qquad \tilde{B}(z,t) = \tilde{B}_o e^{i(kz-\omega t)}$$

Where $\tilde{E}_o$ and $\tilde{B}_o$ are the complex amplitudes, the physical fields are the real parts of $\tilde{E}$ and $\tilde{B}$.

Since $\vec{\nabla}.\vec{\mathrm{E}} = 0$ and $\vec{\nabla}.\vec{\mathrm{B}} = 0$, so it follows that

$$\left(\tilde{E}_o\right)_z = \left(\tilde{B}_o\right)_z = 0$$

This means that electromagnetic waves are transverse, the electric and magnetic fields are perpendicular to the direction of propagation.

Moreover, Faraday's law $\vec{\nabla} \times \vec{\mathrm{E}} = -\frac{\partial \vec{B}}{\partial t}$ implies that:

$$-k\left(\tilde{E}_o\right)_y = \omega\left(\tilde{B}_o\right)_x \quad \text{and} \quad k\left(\tilde{E}_o\right)_x = \omega\left(\tilde{B}_o\right)_y$$

$$\tilde{B}_o = \frac{k}{\omega}\left(\hat{z} \times \tilde{E}_o\right)$$

$\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ are in phase and mutually perpendicular, their real amplitudes are related by:

$$B_o = \frac{k}{\omega}E_o = \frac{1}{c}E_o$$

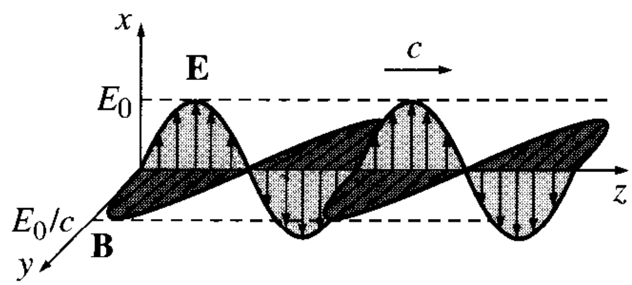
If $\vec{\mathrm{E}}$ point in the x-direction then $\vec{\mathrm{B}}$ points in the y-direction:

$$\tilde{E}(z,t) = \tilde{E}_o e^{i(kz-\omega t)}\hat{\imath} \quad \text{and} \quad \tilde{B}(z,t) = \tilde{B}_o e^{i(kz-\omega t)}\hat{\jmath}$$

Or if we take only the real part, then:

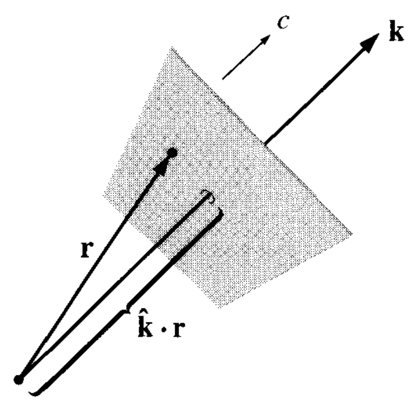
$$\vec{E}(z,t) = E_o \cos(kz - \omega t + \delta)\,\hat{\imath}$$

$$\vec{B}(z,t) = \frac{1}{c} E_o \cos(kz - \omega t + \delta)\,\hat{\jmath}$$



The wave as a whole is said to be polarized along the x-axis (by convention the direction of electric field is used to specify the polarization of an electromagnetic wave.)

If the wave is not propagating along the z-direction instead in any arbitrary direction then vector $\vec{k}$ is the direction of propagation. The scalar product $\vec{k}.\vec{r}$ is the appropriate generalization of kz



$$\tilde{E}(\vec{r},t) = \tilde{E}_o e^{i(\vec{k}.\vec{r} - \omega t)} \hat{n}$$

$$\tilde{B}(\vec{r},t) = \frac{1}{c} \tilde{E}_o e^{i(\vec{k}.\vec{r} - \omega t)} (\hat{k} \times \hat{n}) = \frac{1}{c} \hat{k} \times \vec{E}$$

Where $\hat{n}$ is the polarization vector. Because $\vec{E}$ is transvers, so:

$$\hat{n}.\hat{k} = 0$$

Real electric and magnetic fields in a monochromatic plane wave with propagation vector $\hat{k}$ and polarization $\hat{n}$ are:

$$\vec{E}(\vec{r},t) = E_o \cos(\vec{k}.\vec{r} - \omega t + \delta)\,\hat{n}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c} E_o \cos(\vec{k}.\vec{r} - \omega t + \delta)\,(\hat{k} \times \hat{n}) = \frac{1}{c}(\hat{k} \times \vec{E})$$

9.2.3 Energy and Momentum in Electromagnetic Waves

Energy per unit volume stored in an electromagnetic field is given by:

$$u_{em} = \frac{1}{2}\left(\epsilon_o E^2 + \frac{1}{\mu_o} B^2\right)$$

In the case of monochromatic plane wave:

$$B^2 = \frac{1}{c^2}E^2 = \epsilon_o \mu_o E^2$$

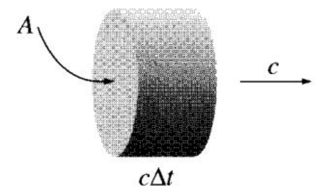
So the electric and magnetic fields are equal:

$$u_{em} = \epsilon_o E^2 = \epsilon_o E_o^2 \cos^2(kz - \omega t + \delta)$$

As the wave travels, it carries this energy along with it. The energy flux density transported by the fields is given by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_o}\left(\vec{E} \times \vec{B}\right)$$

$$\vec{S} = c\epsilon_o E_o^2 \cos^2(kz - \omega t + \delta)\,\hat{k} = cu\hat{k}$$



In a time Δt, a length $c\Delta t$ passes through area A, carrying with an energy $uAc\Delta t$. The energy per unit area per unit time is therefore uc.

Electromagnetic fields not only carry energy they also carry momentum, and the momentum density stored in the field is:

$$\vec{\wp} = \frac{1}{c^2}\vec{S}$$

For monochromatic plane waves:

$$\vec{\wp} = \frac{1}{c}\left(\epsilon_o E_o^2 \cos^2(kz - \omega t + \delta)\right)\hat{k} = \frac{1}{c}u\,\hat{k}$$

In the case of light, the wavelength is so short $(\sim 5 \times 10^{-7} m)$ and the period so brief $(\sim 10^{-15} s)$ that any macroscopic measurement will encompass many cycles. Therefore it is more meaningful to talk about the time average values of energy and momentum over a complete cycle.

$$\langle u \rangle = \frac{1}{2}\epsilon_o E_o^2$$

$$\langle \vec{S} \rangle = \frac{1}{2}c\epsilon_o E_o^2\,\hat{k}$$

$$\langle \vec{\wp} \rangle = \frac{1}{2c}\epsilon_o E_o^2\,\hat{k}$$

The average power per unit area transported by the electromagnetic wave is called the intensity:

$$I \equiv \langle \vec{S} \rangle = \frac{1}{2}c\epsilon_o E_o^2\,\hat{k}$$

When light falls on a perfect absorber it delivers its momentum to the surface. In a time Δt the momentum transfer is:

$$\Delta\vec{p} = \langle\vec{\wp}\rangle Ac\Delta t$$

So the radiation pressure is:

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\epsilon_o E_o^2 = \frac{I}{c}$$

On a perfect reflector the pressure is twice as great because the momentum switches the direction instead of being absorbed.

Example 2:

The intensity of the sunlight hitting the earth is approximately 1300 W/m^2. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

Solution:

$$P = \frac{I}{c} = \frac{1300}{3.0 \times 10^8} = 4.3 \times 10^{-6} N/m^2$$

For a perfect reflector the pressure is twice as much, so:

$$P_{perfect\ Reflector} = 8.6 \times 10^{-6} N/m^2$$

$$\frac{P_{perfect}}{P_{atm}} = \frac{8.6 \times 10^{-6}}{1.03 \times 10^5} = 8.3 \times 10^{-11} atmospheres$$

Example 3:

Find all elements of the Maxwell stress tensor for a monochromatic plane wave travelling in the z–direction and linearly polarized in the x-direction. Does your answer make sense? How is the momentum flux density related to the energy density, in this case?

Solution:

Components of Maxwell's stress tensor are given as:

$$T_{ij} = \epsilon_o\left(E_i E_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_o}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

If the EM wave is linearly polarized in the x-direction, it means electric field is along x-axis and magnetic field is along y-axis and wave is propagating along z-direction.

$$T_{xx} = \epsilon_o\left(E_xE_x - \frac{1}{2}E^2\right) + \frac{1}{\mu_o}\left(B_xB_x - \frac{1}{2}B^2\right) = \frac{1}{2}\left(\epsilon_o E^2 - \frac{1}{\mu_o}B^2\right)$$

For a monochromatic plane wave:

$$B^2 = \frac{1}{c^2}E^2 = \epsilon_o\mu_o E^2$$

$$T_{xx} = \frac{1}{2}\left(\epsilon_o E^2 - \frac{1}{\mu_o}(\epsilon_o\mu_o E^2)\right) = 0$$

$$T_{yy} = \epsilon_o\left(E_yE_y - \frac{1}{2}E^2\right) + \frac{1}{\mu_o}\left(B_yB_y - \frac{1}{2}B^2\right) = \frac{1}{2}\left(-\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right) = 0$$

$$T_{zz} = \epsilon_o\left(E_zE_z - \frac{1}{2}E^2\right) + \frac{1}{\mu_o}\left(B_zB_z - \frac{1}{2}B^2\right) = \frac{1}{2}\left(-\epsilon_o E^2 - \frac{1}{\mu_o}B^2\right) = -u$$

$$T_{zz} = \epsilon_o E_o^2\cos^2(kz - \omega t + \delta)$$

The momentum of these fields is in the z direction, and it is being transported in the z direction, so yes, it does make sense that T_{zz}should be the only nonzero element in T_{ij} .

Since $-\overleftrightarrow{T}.\,da$ is the rate at which momentum crosses an area da. Here we have no momentum crossing areas oriented in the x or y direction;

The momentum per unit time per unit area flowing across a surface oriented in the z direction is:

$$-T_{zz} = u = \wp c$$

$\Delta p = \wp cA\Delta t$; $\frac{\Delta p}{\Delta t} = \wp cA$ = momentum per unit time crossing area A.

9.3 Electromagnetic Waves in Matter

Inside matter but in regions where there are no free charges or free currents, Maxwell's equations are:

$$\vec{\nabla}.\vec{\mathrm{D}} = 0 \quad ; \quad \vec{\nabla} \times \vec{\mathrm{E}} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}.\vec{\mathrm{B}} = 0 \quad ; \quad \vec{\nabla} \times \vec{\mathrm{H}} = \frac{\partial \vec{D}}{\partial t}$$

If the medium is linear and homogeneous (so that ϵ and μ does not vary from point to point),

$$\vec{\mathrm{D}} = \epsilon\vec{E} \quad ; \quad \vec{\mathrm{H}} = \frac{1}{\mu}\vec{B}$$

The Maxwell's equations reduce to:

$$\vec{\nabla}.\vec{\mathrm{E}} = 0 \quad ; \quad \vec{\nabla} \times \vec{\mathrm{E}} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}.\vec{\mathrm{B}} = 0 \quad ; \quad \vec{\nabla} \times \vec{\mathrm{B}} = \mu\epsilon\frac{\partial \vec{E}}{\partial t}$$

The speed of electromagnetic wave through a linear homogeneous medium is:

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}$$

Where $n = \sqrt{\frac{\epsilon\mu}{\epsilon_o\mu_o}}$ is the index of refraction of the material.

For most materials μ is very close to μ_o, hence

$$n \cong \sqrt{\frac{\epsilon}{\epsilon_o}} = \sqrt{\epsilon_r}$$

Where ϵ_r is almost always greater than 1, and hence light travels slowly through matter. All the electrodynamics results can be carried over in materials by simply replacing $\epsilon_o \to \epsilon$ and $\mu_o \to \mu$

The energy density in a homogeneous linear material is:

$$u = \frac{1}{2}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)$$

And the Poynting vector is:

$$\vec{S} = \frac{1}{\mu}\left(\vec{E} \times \vec{B}\right)$$

For a monochromatic plane wave $\omega = kv$ and the amplitude of $\vec{B}$ is $1/v$ times the amplitude of $\vec{E}$ and the intensity is:

$$I = \frac{1}{2}\epsilon v E_o^2$$

When a wave passes from one transparent medium to another medium for example from air to water or glass, then the boundary conditions are as follows:

(i) $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$
(ii) $B_1^{\perp} = B_2^{\perp}$
(iii) $E_1^{\parallel} = E_2^{\parallel}$
(iv) $\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$

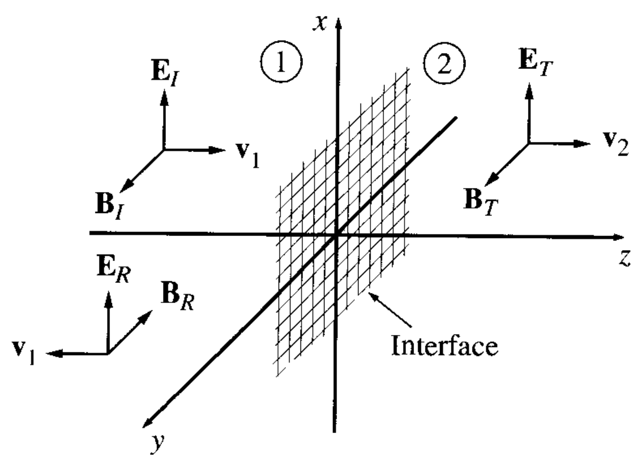
9.3.2 Reflection and Transmission at Normal Incidence

Considering an electromagnetic wave of frequency ω is propagating along z-axis and is polarized along the x-axis. The xy-plane forms the boundary between two linear media.

If the wave is incident on the plane from the left:

$$\tilde{E}_I(z,t) = \tilde{E}_{o_I} e^{i(k_1 z - \omega t)}\hat{\imath}$$

$$\tilde{B}_I(z,t) = \frac{1}{v_1}\tilde{E}_{o_I} e^{i(k_1 z - \omega t)}\hat{\jmath}$$



The electric and magnetic fields of the reflected wave will be:

$$\tilde{E}_R(z,t) = \tilde{E}_{o_R} e^{i(-k_1 z - \omega t)}\hat{\imath}$$

$$\tilde{B}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{o_R} e^{i(-k_1 z - \omega t)}\hat{\jmath}$$

And the electric and magnetic field components of the transmitted beam would be:

$$\tilde{E}_T(z,t) = \tilde{E}_{o_T} e^{i(k_2 z - \omega t)}\hat{\imath}$$

$$\tilde{B}_T(z,t) = \frac{1}{v_2}\tilde{E}_{o_T} e^{i(k_2 z - \omega t)}\hat{\jmath}$$

Note that magnetic field of the reflected wave is in the negative y-direction because

$[\vec{B}(\vec{r},t) = \frac{1}{c}E_o\cos(\vec{k}.\vec{r} - \omega t + \delta)\,(\hat{k}\times\hat{n}) = \frac{1}{c}(-\hat{z}\times\vec{E})]$ and also Poynting vector points in the direction of propagation.

At z=0 the combined fields on the left $[\tilde{E}_I + \tilde{E}_R]$ and $[\tilde{B}_I + \tilde{B}_R]$must join the fields on the right $\tilde{E}_T$ and $\tilde{B}_T$.

Hence the boundary conditions (iii) and (iv) require that fields on both side of the interface are equal:

$$\tilde{E}_{o_I} + \tilde{E}_{o_R} = \tilde{E}_{o_T}$$

$$\frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{o_I} - \frac{1}{v_1}\tilde{E}_{o_R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{o_T}\right)$$

$$\tilde{E}_{o_I} - \tilde{E}_{o_R} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{E}_{o_T} = \beta\tilde{E}_{o_T}$$

Where

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$$

$$\tilde{E}_{o_R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{o_I}\ ;\ \tilde{E}_{o_T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{o_I}$$

In most cases the values of permitivities are close to the values in vacuum ($\mu_1 = \mu_2 = \mu_0$), so

$$\beta = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

$$\tilde{E}_{o_R} = \left(\frac{v_2-v_1}{v_2+v_1}\right)\tilde{E}_{o_I}\ ;\ \tilde{E}_{o_T} = \left(\frac{2v_2}{v_2+v_1}\right)\tilde{E}_{o_I}$$

This result is similar to what we got in the case of wave on a string. The reflected wave is in phase with the incident wave if $v_2 > v_1$ and out of phase if $v_2 < v_1$.

The real amplitudes are related as:

$$E_{o_R} = \left|\frac{v_2-v_1}{v_2+v_1}\right| E_{o_I}\ ;\ E_{o_T} = \left(\frac{2v_2}{v_2+v_1}\right)E_{o_I}$$

$$E_{o_R} = \left|\frac{n_1-n_2}{n_1+n_2}\right| E_{o_I}\ ;\ E_{o_T} = \left(\frac{2n_1}{n_1+n_2}\right)E_{o_I}$$

The intensity of an electromagnetic waves is given by:

$$I = \frac{P_{avg}}{Area} = \frac{1}{2}\epsilon v E_o^2$$

If $\mu_1 = \mu_2 = \mu_o$ then the ratio of the reflected intensity to the incident intensity is:

$$R = \frac{I_R}{I_I} = \left(\frac{E_{o_R}}{E_{o_I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

And the ratio of the transmitted intensity to the incident intensity is:

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{o_T}}{E_{o_I}}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

R is called the reflection coefficient and T is called the transmission coefficient, they measure the fraction of the incident energy that is reflected and transmitted, respectively. And conservation of energy requires that:

$$R + T = 1$$

When light passes from air (n=1) into glass (n=1.5), R=0.04 and T=0.96, this means most of the light is transmitted through the glass and a small portion is reflected.

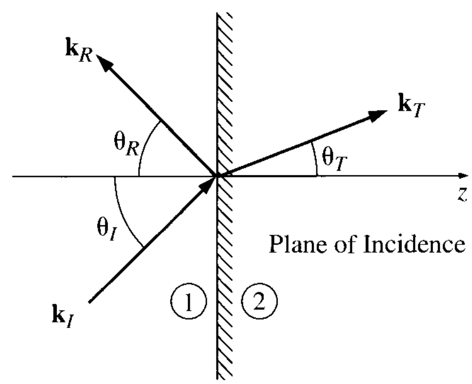
9.3.3 Reflection and Transmission at Oblique incidence

In this case incident wave hits the interface at an incident angle θ_I (normal incidence is a special case when $\theta_I = 0$).

Suppose a monochromatic plane wave approaching the interface from the left is given by:

$$\tilde{E}_I(\vec{r},t) = \tilde{E}_{o_I} e^{i(\vec{k}_I.\vec{r}-\omega t)}$$

$$\tilde{B}_I(\vec{r},t) = \frac{1}{v_1}\left(\hat{k}_I \times \tilde{E}_I\right)$$



The reflected wave is:

$$\tilde{E}_R(\vec{r},t) = \tilde{E}_{o_R} e^{i(\vec{k}_R.\vec{r}-\omega t)}\ ;\ \tilde{B}_R(\vec{r},t) = \frac{1}{v_1}\left(\hat{k}_R \times \tilde{E}_R\right)$$

And a transmitted wave is given by:

$$\tilde{E}_T(\vec{r},t) = \tilde{E}_{o_T} e^{i(\vec{k}_T.\vec{r}-\omega t)}\ ;\ \tilde{B}_T(\vec{r},t) = \frac{1}{v_2}\left(\hat{k}_T \times \tilde{E}_T\right)$$

All three waves have the same frequency ω and the three wavenumbers are related as:

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega \qquad \text{or} \qquad k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

The combined fields in medium (1), $\tilde{E}_I + \tilde{E}_R$ and $\tilde{B}_I + \tilde{B}_R$ are related with the transmitted fields $\tilde{E}_T$ and $\tilde{B}_T$ using the boundary conditions:

(i) $\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}$
(ii) $B_1^{\perp} = B_2^{\perp}$
(iii) $E_1^{\parallel} = E_2^{\parallel}$
(iv) $\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}$

These all share the same generic structure:

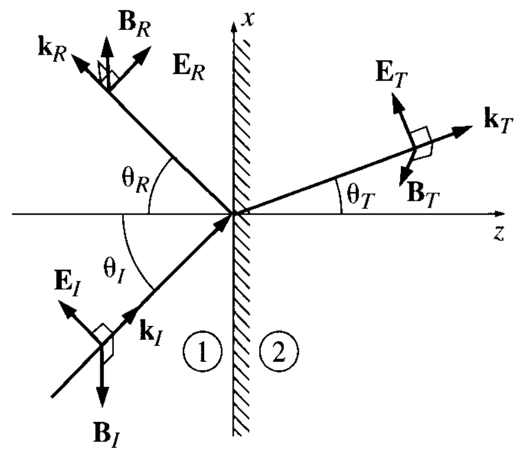
$$(\quad)e^{i(\vec{k}_I.\vec{r}-\omega t)} + (\quad)e^{i(\vec{k}_R.\vec{r}-\omega t)} = (\quad)e^{i(\vec{k}_T.\vec{r}-\omega t)} \quad \text{at } z = 0$$

Notice all x,y and t dependence are in the exponents, and since boundary conditions must hold at all the points on the plane at z=0, so the exponents must be equal, which gives:

$$\vec{k}_I.\vec{r} = \vec{k}_R.\vec{r} = \vec{k}_T.\vec{r} \quad \text{when } z = 0$$

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y$$
$$= x(k_T)_x + y(k_T)_y$$

For all x and all y at z=0.



The above equation holds only if the components separately are equal. For say x=0

$$(k_I)_y = (k_R)_y = (k_T)_y$$

Similarly for y=0

$$(k_I)_x = (k_R)_x = (k_T)_x$$

We can in fact orient our axes such that k_I lies in the xz-plane so that $(k_I)_y = 0 = (k_R)_y = (k_T)_y$

First Law: The incident, reflected and transmitted wave vector form a plane (called **plane of incidence**), which also includes normal to the surface (z-axis in this case).

Also the equation:

$$(k_I)_x = (k_R)_x = (k_T)_x$$

Implies that: $$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

Where θ_I is the angle of incidence and θ_R is the angle of reflection and θ_T is the angle of transmission also called angle of refraction and they are all measured with respect to the normal.

Second Law: The angle of incidence is equal to the angle of reflection.

$$\theta_I = \theta_R \qquad \textbf{Law of Reflection}$$

Third Law: For the transmitted angle

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

This is the **law of refraction** or **Snell's Law**.

These are the three fundamental laws of geometrical optics.

The boundary conditions imply that:

(i) $\epsilon_1\left(\tilde{E}_{o_I} + \tilde{E}_{o_R}\right)_z = \epsilon_2\left(\tilde{E}_{o_T}\right)_z$
(ii) $\left(\tilde{B}_{o_I} + \tilde{B}_{o_R}\right)_z = \left(\tilde{B}_{o_T}\right)_z$
(iii) $\left(\tilde{E}_{o_I} + \tilde{E}_{o_R}\right)_{x,y} = \left(\tilde{E}_{o_T}\right)_{x,y}$

(iv) $\frac{1}{\mu_1}\left(\tilde{B}_{o_I} + \tilde{B}_{o_R}\right)_{x,y} = \frac{1}{\mu_2}\left(\tilde{B}_{o_T}\right)_{x,y}$

Where $\tilde{B}_o = \frac{1}{v}\left(\hat{k} \times \tilde{E}_o\right)$

Suppose the polarization of the incident wave is parallel to the plane of incidence (xz plane in the figure above.) Then the first boundary condition (i) implies that:

$$\epsilon_1\left(-\tilde{E}_{o_I}\sin\theta_I + \tilde{E}_{o_R}\sin\theta_R\right) = \epsilon_2\left(-\tilde{E}_{o_T}\sin\theta_T\right)$$

Since B has no z-component, hence boundary condition (ii) does not add anything. Boundary condition (iii) implies:

$$\tilde{E}_{o_I}\cos\theta_I + \tilde{E}_{o_R}\cos\theta_R = \tilde{E}_{o_T}\cos\theta_T$$

$$\tilde{E}_{o_I} + \tilde{E}_{o_R} = \frac{\cos\theta_T}{\cos\theta_I}\tilde{E}_{o_T} = \alpha\tilde{E}_{o_T}$$

Where $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$

And boundary condition (iv) gives:

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_I} - \tilde{E}_{o_R}\right) = \frac{1}{\mu_2 v_2}\tilde{E}_{o_T}$$

Using the laws of reflection and refraction, we get:

$$\tilde{E}_{o_I} - \tilde{E}_{o_R} = \frac{\mu_1 v_1}{\mu_2 v_2}\tilde{E}_{o_T} = \beta\tilde{E}_{o_T}$$

Where $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \cong \frac{n_2}{n_1}$

$\tilde{E}_{o_R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_I}$ and $\tilde{E}_{o_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_I}$

These are known as **Fresnel's equations** for the case of polarization in the plane of the incidence.

From the above equation we can see that transmitted wave is always in phase with the incidence but the reflected wave is in phase with the incidence wave if $\alpha > \beta$ and will be out of phase by $180°$ if $\alpha < \beta$.

The amplitudes of the transmitted and reflected wave depend on the angle of incidence:

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left[\left(\frac{n_1}{n_2}\right)\sin\theta_I\right]^2}}{\cos\theta_I}$$

In the case of normal incidence $\theta_I = 0$ and $\alpha = 1$

At grazing incidence angel $\theta_I = 90$ and α diverges, which means that the wave is totally reflected.

$$\tilde{E}_{0_R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0_I} = \tilde{E}_{0_I} \quad \text{and} \quad \tilde{E}_{0_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0_I} = 0$$

There is an intermediate angle (θ_B, **Brewster's angle**) where the reflected wave is completely extinguished and this occurs when $\alpha = \beta$.

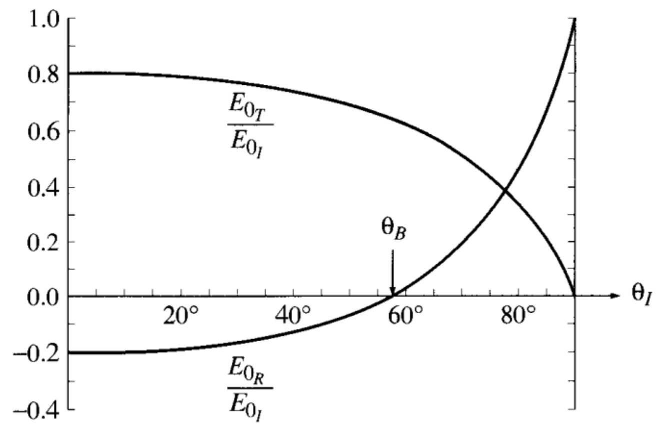
$$\alpha = \frac{\sqrt{1-\left[\left(\frac{n_1}{n_2}\right)\sin\theta_B\right]^2}}{\cos\theta_B} = \beta$$

$$\sin^2\theta_B = \frac{1-\beta^2}{\left(\frac{n_1}{n_2}\right)^2 - \beta^2}$$

Typically $\mu_1 \cong \mu_2$, so $\beta \cong n_2/n_1$

$$\sin^2\theta_B = \frac{\beta^2}{1+\beta^2}$$

$$\tan\theta_B \cong \frac{n_2}{n_1}$$



The figure shows a plot of the reflected and transmitted amplitudes as a function of incidence angle θ_I , for light incident on glass ($n_2 = 1.5$) from air ($n_1 = 1.0$).

The power per unit area striking the surface is $\vec{S}.\hat{z}$, thus the incident intensity is:

$$I_I = \frac{1}{2}\epsilon_1 v_1 E_{0_I}^2 \cos\theta_I$$

While the reflected and transmitted intensities are:

$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0_R}^2 \cos\theta_R$$

$$I_T = \frac{1}{2}\epsilon_2 v_2 E_{0_T}^2 \cos\theta_T$$

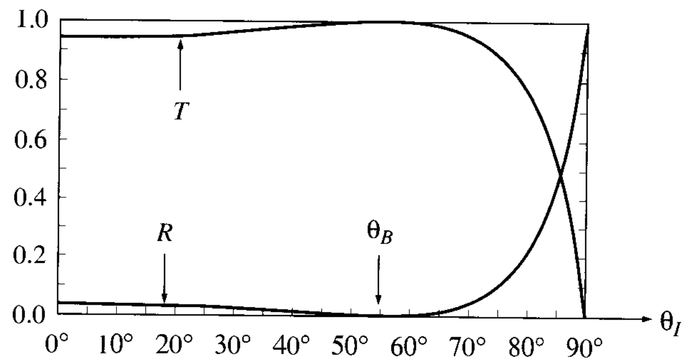
The reflection and transmission coefficients of waves polarized along the plane of incidence are:

$$R = \frac{I_R}{I_I} = \left(\frac{E_{o_R}}{E_{o_I}}\right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{o_T}}{E_{o_I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha + \beta}\right)^2$$

The figure shows a plot of reflection and transmission coefficients as a function of incidence angle (from air/glass interface).

R+T=1 as required by the conservation of energy. At Brewster's angle the refection coefficient is zero whereas transmission coefficient is 1.



Example 4: The index of refraction of diamond is 2.42. Construct the graph of $\frac{E_{o_T}}{E_{o_i}}$ and $\frac{E_{o_R}}{E_{o_I}}$ vs. θ_I (incidence angle) for the air/diamond interface. (Assume $\mu_1 = \mu_2 = \mu_o$). Calculate:

a) The amplitudes at normal incidence.
b) Brewster's angle
c) The "crossover" angle at which the reflected and transmitted amplitudes are equal.

Solution:

a)

$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{n_2}{n_1} = \frac{2.42}{1} = 2.42$$

$$\alpha = \frac{\sqrt{1 - \sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1 - \left[\left(\frac{n_1}{n_2}\right)\sin\theta_I\right]^2}}{\cos\theta_I} = \frac{\sqrt{1 - \left[\frac{1}{2.42}\sin 0\right]}}{\cos 0} = 1$$

$$\tilde{E}_{o_R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_I}\,,\ \tilde{E}_{o_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_I}$$

$$\frac{\tilde{E}_{o_R}}{\tilde{E}_{o_I}} = \frac{1 - 2.42}{1 + 2.4} = -0.415$$

$$\frac{\tilde{E}_{o_T}}{\tilde{E}_{o_I}} = \frac{2}{1 + 2.4} = 0.585$$

b)

$$\tan\theta_B = \frac{n_2}{n_1} = 2.42$$

$$\theta_B = \tan^{-1}(2.42) = 67.5°$$

c)

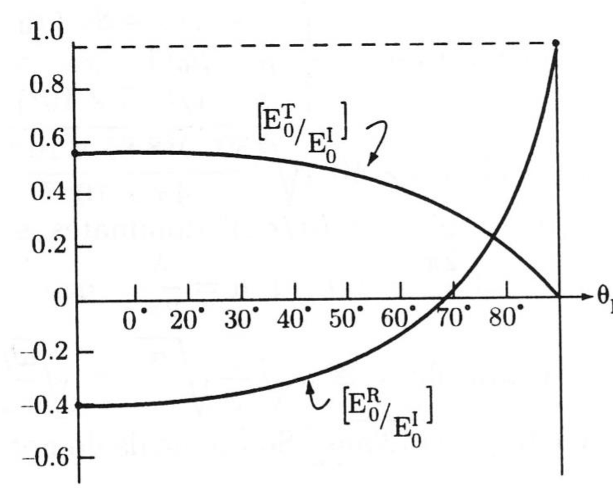
$$\tilde{E}_{o_R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{o_I}\ ,\ \tilde{E}_{o_T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{o_I}$$

$$\tilde{E}_{o_R} = \tilde{E}_{o_T}$$

$$\alpha - \beta = 2 \rightarrow \alpha = \beta + 2 = 4.42$$

$$\alpha^2 = \frac{1 - \left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_I}{\cos^2\theta_I}$$

$$\theta_I = 78.3°$$



9.4 Absorption and Dispersion

9.4.1 Electromagnetic Waves in Conductors

In conductors, the free charge density and hence the free current density is not zero. According to Ohm's law, the free current density in a conductor is proportional to the applied electric field,

$$\vec{J}_f = \sigma\vec{E}$$

Hence, Maxwell's equation for linear media assume the form:

$$\text{(i)} \qquad \vec{\nabla}.\vec{\mathrm{E}} = \frac{1}{\epsilon}\rho_f$$

$$\text{(ii)} \qquad \vec{\nabla}.\vec{\mathrm{B}} = 0$$

$$\text{(iii)} \qquad \vec{\nabla}\times\vec{\mathrm{E}} = -\frac{\partial\vec{B}}{\partial t}$$

$$\text{(iv)} \qquad \vec{\nabla}\times\vec{\mathrm{B}} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

Now the continuity equation for the free charge is:

$$\vec{\nabla}.\vec{\mathrm{J}}_f = -\frac{\partial\rho_f}{\partial t}$$

Together with Ohm's law and Gauss's law:

$$\frac{\partial\rho_f}{\partial t} = -\sigma\left(\vec{\nabla}.\vec{E}\right) = -\frac{\sigma}{\epsilon}\rho_f$$

For a homogeneous linear medium,

$$\rho_f(t) = e^{\left(-\frac{\sigma}{\epsilon}\right)t}\rho_f(0)$$

Thus any initial free charge density will dissipate in a characteristic time $\tau = \epsilon/\sigma$.

This reflects the fact that if we have some free charge on a conductor, it will flow out to the edges. For a good conductor $\sigma = \infty$ and $\tau = 0$, which means that τ is much less than any other relevant times. For an oscillatory system it would mean that $\tau \ll 1/\omega$ and for a poor conductor $\gg \omega$.

After disappearance of any accumulated charge on the conductor:

$$\text{(i)} \qquad \vec{\nabla}.\vec{\mathrm{E}} = 0$$

$$\text{(ii)} \qquad \vec{\nabla}.\vec{\mathrm{B}} = 0$$

$$\text{(iii)} \qquad \vec{\nabla}\times\vec{\mathrm{E}} = -\frac{\partial\vec{B}}{\partial t}$$

$$\text{(iv)} \qquad \vec{\nabla}\times\vec{\mathrm{B}} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

If we apply curl to last two equations above, we get:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathrm{E}}) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B})$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathrm{E}}) = \vec{\nabla}(\vec{\nabla}.\vec{\mathrm{E}}) - \nabla^2\vec{\mathrm{E}} = -\nabla^2\vec{\mathrm{E}} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\frac{\partial}{\partial t}\left(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t}\right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{\mathrm{B}}) = \mu\sigma(\vec{\nabla} \times \vec{E}) + \mu\epsilon\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})$$

$$\nabla^2\mathbf{E} = \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{E}}{\partial t}, \quad \nabla^2\mathbf{B} = \mu\epsilon\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu\sigma\frac{\partial\mathbf{B}}{\partial t}.$$

These equations still give plane-wave solutions:

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)}$$

But this time $\tilde{k}$ is complex:

$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$ or $\tilde{k} = k + i\kappa$

Where

$$k = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{1/2}$$

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{1/2}$$

The imaginary part of $\tilde{k}$ results in an attenuation of the wave (decreasing amplitude with increasing z):

$$\tilde{E}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}$$

$$\tilde{B}(z,t) = \tilde{B}_o e^{-\kappa z} e^{i(kz-\omega t)}$$

The distance it takes to reduce the amplitude by a factor of $1/e$ (about a third) is called the **skin depth**.

$$d \equiv \frac{1}{\kappa}$$

This is a measure of how far a wave can penetrate into a conductor. And the real part of $\tilde{k}$ determines the wavelength, the propagation speed and index of refraction:

$\lambda = \frac{2\pi}{k}$ and $v = \frac{\omega}{k}$ and $n = \frac{ck}{\omega}$

Now let's consider $\vec{\boldsymbol{E}}$ is polarized along x-axis, then:

$$\tilde{\boldsymbol{E}}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)} \hat{\boldsymbol{\imath}}$$

$$\tilde{\boldsymbol{B}}(z,t) = \frac{\tilde{k}}{\omega} \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)} \hat{\boldsymbol{\jmath}}$$

Like any complex number $\tilde{k}$ can be expressed as:

$$\tilde{k} = K e^{i\phi}$$

$$K = |\tilde{k}| = \sqrt{k^2 + \kappa^2} = \omega \sqrt{\epsilon\mu\left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right)}$$

$$\phi = \tan^{-1}(\kappa/k)$$

We can write the complex amplitudes of the electric and magnetic field as:

$$\tilde{E}_o = E_o e^{i\delta_E} \quad \text{and} \quad \tilde{B}_o = B_o e^{i\delta_B}$$

$$B_o e^{i\delta_B} = \frac{K e^{i\phi}}{\omega} E_o e^{i\delta_E}$$

Evidently, the electric and magnetic field are no longer in phase, and

$$\delta_B - \delta_E = \phi$$

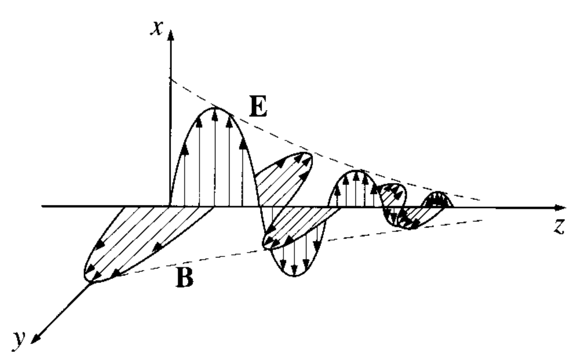
The magnetic field lags behind the electric field, and the real amplitudes of **E** and **B** are related by:

$$\frac{B_o}{E_o} = \frac{K}{\omega} = \sqrt{\epsilon\mu\left(\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2}\right)}$$

The real electric and magnetic fields are:

$$\boldsymbol{E}(z,t) = E_o e^{-\kappa z} \cos(kz - \omega t + \delta_E)\, \hat{\boldsymbol{\imath}}$$

$$\boldsymbol{B}(z,t) = B_o e^{-\kappa z} \cos(kz - \omega t + \delta_E + \phi)\, \hat{\boldsymbol{\jmath}}$$



Example 5:

(a) Suppose some free charge is embedded in a piece of glass (n=1.5). About how long would it take for the charge to flow to the surface. Consider the conductivity of the glass to be $(\sigma = \frac{1}{\rho} = 10^{-12} (\Omega.m)^{-1}$.

(b) Silver is an excellent conductor with resistivity $\rho = 1.59 \times 10^{-8}\ \Omega.m$ but it is expensive. To design a microwave experiment that can operate a frequency of 10^{10} Hz, how thick should be the silver coatings.

(c) Find the wavelength and propagation speed in copper for radio waves at 1 MHz. Compare the corresponding values in air or vacuum.

Solution:

Example 6:

(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$) is $\left(\frac{2}{\sigma}\right)\sqrt{\epsilon/\mu}$ (independent of frequency). Find the skin depth (in meters) for pure water.
(b) Show that the skin depth in a good conductor ($\sigma \gg \omega\epsilon$) is $\lambda/2\pi$ (where λ is the wavelength in the conductor). Find the skin depth (in nanometers) for a typical metal ($\sigma \approx 10^7\ (\Omega.m)^{-1}$) in the visible range ($\omega \approx 10^{15}\ s^{-1}$), assuming $\epsilon \approx \epsilon_o$ and $\mu \approx \mu_o$. Why are metal opaque?
(c) Show that in a good conductor the magnetic field lags the electric field by 45° and fund the ratio of their amplitudes. Fir a numerical example, use the typical metal in part (b).

9.4.2 Reflection at a conducting surface

The boundary conditions between two surfaces that involve free charges are as follows:

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$$

$$B_1^{\perp} - B_2^{\perp} = 0$$

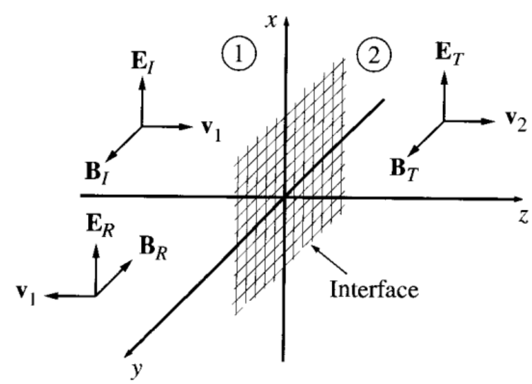
$$E_1^{\parallel} - E_2^{\parallel} = 0$$

$$\frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_f \times \hat{n}$$

Where σ_f is the free surface charge density and $\vec{K}_f$ is the free surface current and $\hat{n}$ is the unit vector perpendicular to the surface, pointing from medium (2) into medium (1).

For Ohmic conductors $\vec{J}_f = \sigma\vec{E}$ (here σ is the conductivity), there can be no surface current because this will require infinite electric field at the boundary.

Suppose now that xy-plane forms the boundary between a nonconducting linear medium (1) and a conductor (2). A monochromatic plane wave travelling in the z-direction and polarized in the x-direction, approaches from the left as shown in the figure.



$$\tilde{E}_I(z,t) = \tilde{E}_{o_I} e^{i(k_1 z - \omega t)} \hat{\imath}$$

$$\tilde{B}_I(z,t) = \frac{1}{v_1} \tilde{E}_{o_I} e^{i(k_1 z - \omega t)} \hat{\jmath}$$

The reflected waves will be:

$$\tilde{E}_R(z,t) = \tilde{E}_{o_R} e^{i(-k_1 z - \omega t)} \hat{\imath} \quad ; \quad \tilde{B}_I(z,t) = -\frac{1}{v_1} \tilde{E}_{o_R} e^{i(-k_1 z - \omega t)} \hat{\jmath}$$

And the transmitted wave in the conductor will be:

$$\tilde{E}_T(z,t) = \tilde{E}_{o_T} e^{i(\tilde{k}_2 z - \omega t)} \hat{\imath} \quad ; \quad \tilde{B}_T(z,t) = \frac{1}{v_2} \tilde{E}_{o_T} e^{i(\tilde{k}_2 z - \omega t)} \hat{\jmath} = \frac{\tilde{k}_2}{\omega} \tilde{E}_{o_T} e^{i(\tilde{k}_2 z - \omega t)} \hat{\jmath}$$

Now applying the boundary condition, since $E_1^{\perp} = E_2^{\perp} = 0$, hence it means that $\sigma_f = 0$.

Third boundary condition $E_1^{\parallel} - E_2^{\parallel} = 0$ will yield:

$$\tilde{E}_{o_I} + \tilde{E}_{o_R} = \tilde{E}_{o_T}$$

And fourth BC with $\vec{K}_f = 0$ will yield:

$$\frac{1}{\mu_1 v_1}\left(\tilde{E}_{o_I} - \tilde{E}_{o_R}\right) - \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{o_T} = 0$$

$$\tilde{E}_{o_I} - \tilde{E}_{o_R} = \frac{\mu_1 v_1 \tilde{k}_2}{\mu_2 \omega}\tilde{E}_{o_T} = \tilde{\beta}\tilde{E}_{o_T}$$

Where

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$$

$$\tilde{E}_{o_R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{o_I}$$

$$\tilde{E}_{o_T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{o_I}$$

These results appear similar to the one that apply at the boundary between nonconductors but here $\tilde{\beta}$ is a complex number.

For a perfect conductor ($\sigma = \infty$) and $\tilde{k}_2 = \infty$ hence $\tilde{\beta} = \infty$ and hence

$$\tilde{E}_{o_R} = -\tilde{E}_{o_I} \quad ; \quad \tilde{E}_{o_T} = 0$$

Remember:

$$\tilde{k} = k + i\kappa, \tag{9.125}$$

where

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \quad \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}. \tag{9.126}$$

In the case of a perfect conductor the incoming EM wave is completely reflected with 180° phase shift. This is why a mirror is made by coating thin layer of silver onto a glass and the light is reflected perfectly from the silver coating. Since the skin depth in silver for optical frequency is ~100Å, so we don't need a thick layer of silver anyway.

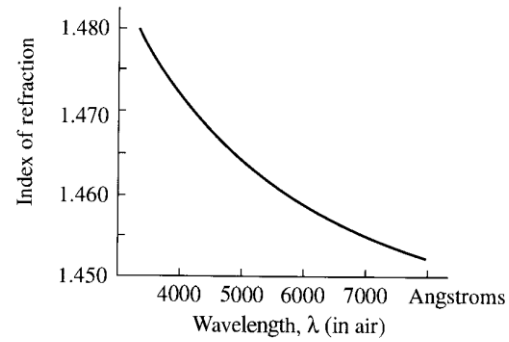
Example 7:

Calcualte the reflection coefficient for light at an air-to-silver interface ($\mu_1 = \mu_2 = \mu_o$ and $\epsilon_1 = \epsilon_o$ and $\sigma = 6 \times 10^7\ (\Omega.m)^{-1}$) at optical frequencies ($\omega = 4 \times 10^{15}/s$)

9.4.3 The frequency dependence of Permittivity

So far, we have considered permittivity, permeability and conductivity as constants but in reality these parameters to some extent depend on the frequency of the waves we are considering.

We know from experience that a prism or raindrop bends blue light more sharply than the red light and spreads white light into a rainbow of colors. This phenomenon is called **dispersion** and ***when the speed of a wave depends on its frequency***, the medium is called **dispersive**.

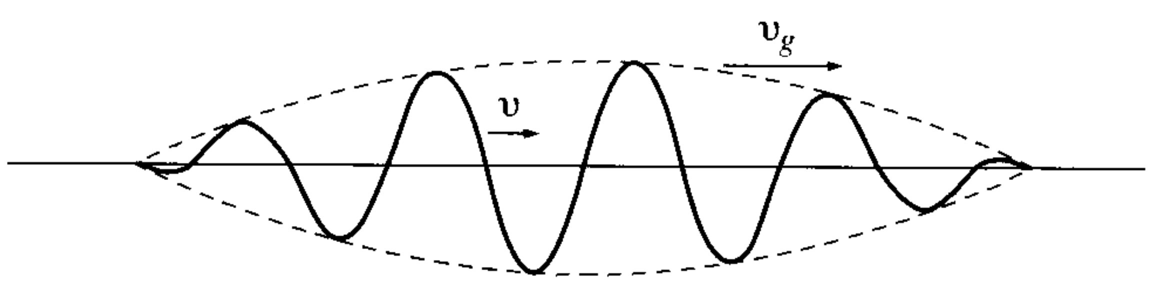


Because of waves of different frequencies travel at different speeds in a dispersive medium, a waveform that incorporates a range of frequencies will change shape as it propagates.

A sharply peaked wave typically flattens out and whereas each sinusoidal components travels at the ordinary **wave velocity** (**phase velocity**).

$$v = \frac{\omega}{k}$$

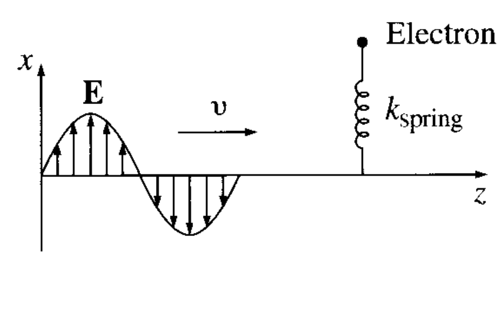
The packet as a whole (the "envelope") travels at the **group velocity** v_g. $v_g = \frac{d\omega}{dk}$



The energy carried out by a wave packet in a dispersive medium ordinarily travels at the group velocity not the phase velocity.

Now lets' consider the frequency dependence of ϵ in non-conductors. Considering the electrons in non-conductors are bound to specific molecules, we can consider this force to be spring force:

$$F_{binding} = -k_{spring}x = -m\omega_o^2 x$$



Where x is the displacement from the equilibrium and m is the mass of the electron and ω_o is the natural oscillation frequency.

Any binding force can be approximated this way for sufficiently small displacements from equilibrium. By expanding the potential energy in a Taylor series about the equilibrium point:

$$U(x) = U(0) + xU'(0) + \frac{1}{2}x^2U''(0) + \cdots$$

The first term is constant and we can adjust the zero of the potential energy so that $U(0) = 0$. The second term is zero because $\frac{dU}{dx} = -F$ which is zero at equilibrium position for spring-mass like forces. The third term however is the potential energy of a spring with spring constant:

$$k_{spring} = \left.\frac{d^2U}{dx^2}\right|_{x=0}$$

For small displacements, the higher terms in the series can be ignored.

There will be some damping force on the bound electron:

$$F_{damping} = -m\gamma\frac{dx}{dt} \quad \text{[damping force is opposite to velocity]}$$

An oscillating charge radiates and the energy is lost due to radiation emission.

In the presence of EM wave of frequency ω, polarized in the x-direction, the electron is subject to driving force, given as:

$$F_{driving} = qE = qE_o\cos(\omega t)$$

Where q is the charge of the electron and E_o is the amplitude of the wave at a point z where the electron is located.

Now using Newton's second law:

$$m\frac{d^2x}{dt^2} = F_{tot} = F_{binding} + F_{damping} + F_{driving}$$

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_o^2x = qE_o\cos(\omega t)$$

We can write a similar equation in complex form and real part of it will be exactly the above equation.

$$\frac{d^2\tilde{x}}{dt^2} + \gamma\frac{d\tilde{x}}{dt} + \omega_o^2\tilde{x} = \frac{q}{m}E_oe^{-i\omega t}$$

In the steady-state the system oscillates at the driving frequency:

$$\tilde{x}(t) = \tilde{x}_oe^{-i\omega t}$$

Using this in the above equation we get:

$$\tilde{x}_o = \frac{q/m}{\omega_o^2 - \omega^2 - i\gamma\omega} E_o$$

The dipole moment is the real part of:

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_o^2 - \omega^2 - i\gamma\omega} E_o e^{-i\omega t}$$

The imaginary term in the denominator means that p is out of phase with E (lagging behind by an angle:

$$\tan^{-1}\left(\frac{\gamma\omega}{\omega_o^2 - \omega^2}\right)$$

which is small for $\omega \ll \omega_o$ and rises to π when $\omega \gg \omega_o$.

In general, differently situated electrons within a given molecule experience different natural frequencies and damping coefficients.

Consider there are f_j electrons with frequency ω_j and damping γ_j in each molecule. If there are N molecules per unit volume, the polarization $\vec{P}$ is given by the real part of:

$$\tilde{P} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{E}$$

And

$$\tilde{P} = \epsilon_o \tilde{\chi}_e \tilde{E}$$

Where $\tilde{\chi}_e$ is the complex electrical susceptibility.

The proportionality between $\tilde{D}$ and $\tilde{E}$ is the **complex permittivity**, like $\tilde{D} = \tilde{\epsilon}\tilde{E}$

$$\tilde{\epsilon} = \epsilon_o(1 + \tilde{\chi}_e)$$

The **complex dielectric constant** is:

$$\tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_o} = 1 + \frac{Nq^2}{m\epsilon_o}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

In a dispersive medium the wave equation for a given frequency is:

$$\nabla^2 \tilde{E} = \tilde{\epsilon}\mu_o \frac{\partial^2 \tilde{E}}{\partial t^2}$$

And the solution of this differential equation is:

$$\tilde{E}(z,t) = \tilde{E}_o e^{i(\tilde{k}z - \omega t)}$$

Where complex wavenumber $\tilde{k}$:

$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu_o}\omega$$

$$\tilde{k} = k + i\kappa$$

$$\tilde{E}(z,t) = \tilde{E}_o e^{-\kappa z} e^{i(kz-\omega t)}$$

The quantity

$$\alpha = 2\kappa$$

Is called the **absorption coefficient**. The wave velocity is ω/k and the index of refraction is:

$$n = \frac{ck}{\omega}$$

Here k and κ are different than what we used before in the electromagnetic waves in conductors, because here they have nothing to do with the conductivity, rather they are determined by the parameters of damped harmonic oscillator.

$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu_o}\omega = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} = \frac{\omega}{c}\left[1 + \frac{Nq^2}{m\epsilon_o}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]^{1/2}$$

$\tilde{k} \cong \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_o}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right]$ using Binomial expansion (ignoring higher order terms)

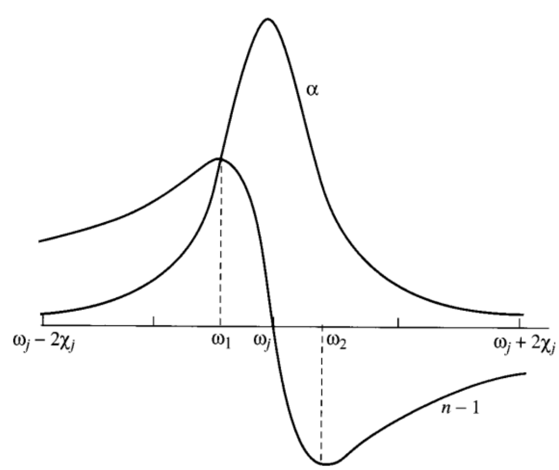
$$n = \frac{ck}{\omega} \cong 1 + \frac{Nq^2}{2m\epsilon_o}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

And

$$\alpha = 2\kappa \cong \frac{Nq^2\omega^2}{m\epsilon_o c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}$$

Mostly, the index of refraction n rises gradually with the increasing frequency. However, in the immediate vicinity of the resonance frequency the index of refraction drops sharply. Because this behavior is atypical, so it is called **anomalous dispersion**.

As can be seen from the figure below that region of anomalous dispersion ($\omega_1 < \omega < \omega_2$) coincides with the region of high absorption.



The material in fact is opaque in this frequency range. The reason is that electrons are being driven at their resonant frequencies and they have large amplitude of oscillations and hence large energy dissipation by damping mechanism.

In the figure, we can also notice that $n < 1$ above the resonant frequency (ω_j), which means that wave speed exceed c. But the energy does not travel at wave speed but the group velocity and also this is an approximation, here the graph does not include other terms in the sum, which add a relatively constant background and keep $n > 1$ on both sides of the resonance.

If the EM wave frequency is away from the resonance, then damping can be ignored and hence:

$$n = 1 + \frac{Nq^2}{2m\epsilon_o}\sum_j \frac{f_j}{\omega_j^2 - \omega^2}$$

For transparent materials, the nearest significant resonances typically lie in the Ultraviolet, so $\omega < \omega_j$

$$\frac{1}{\omega_j^2 - \omega^2} = \frac{1}{\omega_j^2}\left(1 - \frac{\omega^2}{\omega_j^2}\right)^{-1} \cong \frac{1}{\omega_j^2}\left(1 + \frac{\omega^2}{\omega_j^2}\right)$$

$$n = 1 + \left(\frac{Nq^2}{2m\epsilon_o}\sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2\left(\frac{Nq^2}{2m\epsilon_o}\sum_j \frac{f_j}{\omega_j^4}\right)$$

And in terms of wavelength in vacuum ($\lambda = 2\pi c/\omega$):

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right)$$

This is known as **Cauchy's formula**, the constant A is called the **coefficient of refraction** and B is called the **coefficient of dispersion**.

Example 8:

A primitive model of an atom consists of a point nucleus (+q) surrounded by a uniformly charged spherical cloud (-q) of radius *a*.

a) what is the natural frequency of an atom with such a primitive model.
b) Where in the electromagnetic spectrum does this lie assuming the radius of the atom is 0.5 Å?
c) Find the coefficient of refraction and dispersion and compare them with those for hydrogen at 0 °C and atmospheric pressure.

Solution:

(a) The atomic model is like a dipole and electric field due to a dipole is:

$$E = \frac{1}{4\pi\epsilon_o}\frac{qd}{a^3}$$

Force on an electron in this electric field would be:

$$F = -qE = -\left(\frac{1}{4\pi\epsilon_o}\frac{q^2}{a^3}\right)x = -k_{spring}x = -m\omega_o^2 x$$

$$\omega_o = \sqrt{\frac{q^2}{4\pi\epsilon_o m a^3}}$$

(b)

$$\nu = \frac{\omega_o}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{(1.6\times10^{-19})^2}{4\pi(8.85\times10^{-12})(9.11\times10^{-31})(0.5\times10^{-10})^3}} = 7.16\times10^{15}Hz$$

(This is ultraviolet frequency)

(c) The coefficient of refraction is given by:

$$A = \left(\frac{Nq^2}{2m\epsilon_o}\sum_j \frac{f_j}{\omega_j^2}\right)$$

$$N = \#\ of\ molecules\ per\ unit\ volume = \frac{Avogadro's\ number}{22.4\ litres} = \frac{6.02\times10^{23}}{22.4\times10^{-3}}$$

$$= 2.69\times10^{25}\frac{\#}{m^3}$$

$$A = \frac{Nq^2}{2m\epsilon_o}\frac{f}{\omega_o^2} = \frac{2.69\times10^{25}\times(1.6\times10^{-19})^2 * 1}{2 * 9.11\times10^{-31} * 8.85\times10^{-12} * 4\pi^2(7.16\times10^{15})^2} = 4.2\times10^{-5}$$

This is about 1/3 the actual value.

$$B = \left(\frac{2\pi c}{\omega_o}\right)^2 = \left(\frac{2\pi * 3 * 10^8}{2\pi * 7.16 \times 10^{15}}\right)^2 = 1.8 * 10^{-15} m^2$$

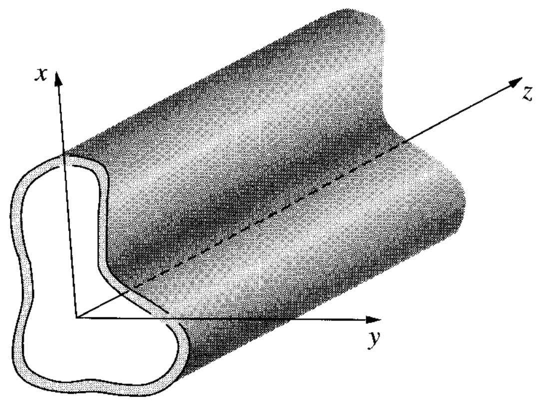
This is about 1/4 the actual value. So even this crude model gets very close to the actual value.

9.5 Guided Waves

9.5.1 Wave Guides

We can confine the electromagnetic waves in the interior of a hollow pipe which can be called waveguide. We will assume that waveguide is a perfect conductor so that $\vec{E} = 0$ and $\vec{B} = 0$ inside the material itself and hence the boundary conditions on the inner walls are:

$$\left.\begin{aligned} &\text{(i)}\ \ E^{\parallel} = 0 \\ &\text{(ii)}\ \ B^{\perp} = 0 \end{aligned}\right\} \quad (1)$$



We will consider the electromagnetic waves that propagate down the tube have the generic form of $\vec{E}$ and $\vec{B}$:

$$\tilde{E}(x, y, z, t) = \tilde{E}_o(x, y)e^{i(kz-\omega t)} \ldots (2)$$

$$\tilde{B}(x, y, z, t) = \tilde{B}_o(x, y)e^{i(kz-\omega t)} \ldots (3)$$

Here, k is real because the wave propagates in the hollow part of the tube. The electric and magnetic fields must satisfy Maxwell's equations in the interior of the wave:

$$\left.\begin{aligned} &\text{(i)} \quad \vec{\nabla}.\vec{E} = 0 \qquad \text{(iii)} \quad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ &\text{(ii)} \quad \vec{\nabla}.\vec{B} = 0 \qquad \text{(iv)} \quad \vec{\nabla}\times\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \end{aligned}\right\} \quad (4)$$

Inside the material of the conducting wave guide, hence

$$\text{(i)} \ \ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = 0 \ \rightarrow \oint \vec{E}.d\vec{l} = -\frac{d}{dt}\int \vec{B}.d\vec{a} = 0 \rightarrow E^{\parallel} = 0 \ \ (\text{since } E^{\parallel}_{inside} = 0)$$

$$\text{(ii)} \ \ \vec{\nabla}.\vec{B} = 0 \ \rightarrow \oint \vec{B}.d\vec{a} = 0 \rightarrow B^{\perp} = 0 \quad (\text{since } B^{\perp}_{inside} = 0)$$

We need to find the electric and magnetic fields which are not in general transverse, to meet the boundary conditions, so we will include the longitudinal components:

$$\tilde{E}_o = E_x\hat{\imath} + E_y\hat{\jmath} + E_z\hat{z} \ \text{ and } \ \tilde{B}_o = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{z} \ \text{-------}(5)$$

Where each of the component is a function of x and y.

Using these in Maxwell's equations (iii) and (iv), we get:

$$\left.\begin{aligned} &\text{(i)} \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z && \text{(iv)} \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z \\ &\text{(ii)} \quad \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x && \text{(v)} \quad \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x \\ &\text{(iii)} \quad ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y && \text{(vi)} \quad ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y \end{aligned}\right\} \quad (6)$$

Equations (ii), (iii), (v) and (vi) can be solved for E_x, E_y, B_x and B_y:

$$\left.\begin{aligned}
E_x &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right) \\
E_y &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right) \\
B_x &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right) \\
B_y &= \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)
\end{aligned}\right\} \quad (7)$$

Using equations (7) into other Maxwell's equations, yields, uncoupled longitudinal components of electric and magnetic field:

$$\left.\begin{aligned}
&\text{(i)} \quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right]E_z = 0 \\
&\text{(ii)} \quad \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right]B_z = 0
\end{aligned}\right\} \quad (8)$$

If $E_z = 0$, we call this transverse electric **(TE)** waves, and if $B_z = 0$ they are called transverse magnetic **(TM)** waves. And if both $E_z = B_z = 0$, then they are called transverse electric and magnetic **(TEM)** waves.

TEM waves cannot occur in a hollow waveguide. Because if $E_z = B_z = 0$ then from equations (6):

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \rightarrow \vec{\nabla}.\vec{\mathrm{E}} = 0$$

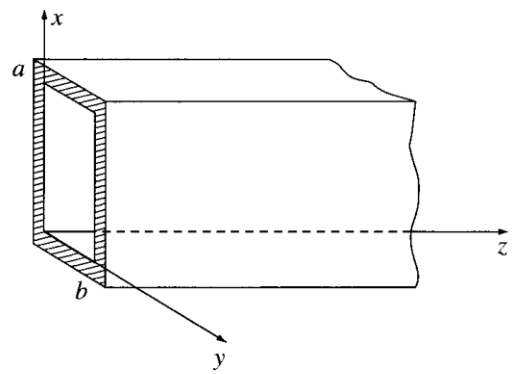
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \rightarrow \vec{\nabla} \times \vec{\mathrm{E}} = 0$$

If the divergence and curl of electric field is zero then $\vec{\mathrm{E}} = -\vec{\nabla}V_{scalar}$ and $\vec{\nabla}.\vec{\mathrm{E}} = \nabla^2 V_{scalar} = 0$

From the boundary condition $E^{\parallel} = 0$ on the inner surface of the wave guide, means that the inner surface of the waveguide is equipotential, i.e. $V_{scalar} = constant$

Since Laplace's equation does not allow any local minima or maxima, hence it means that potential is same everywhere in the waveguide or $\vec{\mathrm{E}} = 0$ in the waveguide i.e. no EM wave in the waveguide.

9.5.2 TE Waves in a Rectangular Waveguide:



Suppose a uniform perfectly conducting hollow rectangular waveguide of inner height *a* and width *b*.

TE waves are propagating in the waveguide:

$E_z = 0$ and $B_z \neq 0$

To solve for Bz, let's use the separation of variables:

$$B_z(x, y) = X(x)Y(y)$$

So that equation (8) becomes:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] X(x)Y(y) = 0$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] XY = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \rightarrow X(x) = A\sin(k_x x) + B\cos(k_x x) \ldots (9)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \rightarrow Y(y) = C\sin(k_y y) + D\cos(k_y y) \ldots (10)$$

$$-k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 - k^2 = 0 \ldots (11)$$

The boundary condition $(B^{\perp} = 0)$ requires that $B_x = 0$ at x=0 and x=*a* and also from equation (7)(iii) in the previous section $\frac{dX}{dx} = 0$ at x=0 and x=*a*.

Equation (9) at x=0 would give $B = 0$:

$$X(x) = A\sin(k_x x)$$

And at x=*a* the above equation gives:

$$A\sin(k_x a) = 0 \rightarrow k_x = \frac{m\pi}{a} \ (m = 0,1,2, \ldots)$$

Similarly for Y,

$$k_y = \frac{n\pi}{b} \; (n = 0,1,2,\dots)$$

$$B_z(x,y) = X(x)Y(y) = B_o \cos\left(m\pi \frac{x}{a}\right) \cos\left(n\pi \frac{y}{b}\right)$$

This solution is called the TE_{mn} mode (the first index is associated with the larger dimension so $a \geq b$) and at least one of the indices must be non-zero.

From equation (11):

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]}$$

If $\omega < c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \equiv \omega_{mn}$

Then the wavenumber is imaginary and instead of travelling wave we get exponentially attenuated field. For this reason ω_{mn} is called **cutoff frequency** for the mode in question. The lowest cutoff frequency for a given waveguide occurs for the mode TE_{10}:

$$\omega_{10} = c\pi\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{c\pi}{a}$$

A wave with frequency less than this will not propagate at all.

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}}$$

Which is greater than c, however the energy carried by the wave travels at the group velocity:

$$v_g = \frac{d\omega}{dk} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c$$

Example 9:

(a) Show that the mode TE_{00} cannot occur in a rectangular waveguide.
(b) Consider a rectangular waveguide with dimensions 2.28cm x 1.01 cm. What TE modes will propagate in this waveguide, if the driving frequency is 1.70×10^{10} Hz?
(c) If you want to excite only one TE mode, what range of frequencies (and corresponding wavelengths in open space) could you use?

Solution:

(a) TE mode implies that $E_z = 0$, and for $m = n = 0$

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} = \frac{\omega}{c}$$

$$\left.\begin{array}{llll}
\text{(i)} & \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z & \text{(iv)} & \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z \\
\text{(ii)} & \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x & \text{(v)} & \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x \\
\text{(iii)} & ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y & \text{(vi)} & ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y
\end{array}\right\} \quad (6)$$

And equation 6(ii) with $E_z = 0$ becomes:

$$E_y = -\frac{\omega}{k}B_x = -cB_x$$

And equation 6(iii) becomes:

$$E_x = (\omega/k)B_y = cB_y$$

And equation 6(v) becomes:

$$\frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x = -\frac{i\omega}{c^2}cB_y = -\frac{i\omega}{c}B_y = -ikB_y$$

$$\frac{\partial B_z}{\partial y} = 0$$

And equation 6(vi) becomes:

$$ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y = -\frac{i\omega}{c^2}(-cB_x) = \frac{i\omega}{c}B_x = ikB_x$$

$$\frac{\partial B_z}{\partial x} = 0$$

Since B_z is a function of x and y only, so it means B_z is constant.

According to Faraday's law:

$$\oint \vec{E}\,.\,d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}.\,d\vec{a}$$

As $\tilde{B}(x,y,z,t) = \tilde{B}_o(x,y)e^{i(kz-\omega t)}$, so:

$$\frac{\partial \vec{B}}{\partial t} = -i\omega\vec{B}$$

$$\oint \vec{E}\,.\,d\vec{l} = i\omega \int \vec{B}.\,d\vec{a}$$

Applying to the cross-section of the waveguide:

$$\oint \vec{E}\,.\,d\vec{l} = i\omega e^{i(kz-\omega t)} \int B_z da = i\omega B_z e^{i(kz-\omega t)}(ab)$$

Since the boundary goes inside the metal where $\vec{E} = 0$ so this makes $B_z = 0$, so this would be TEM mode which we know cannot exist inside the waveguide.

(b)

$$\omega_{mn} = c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\omega_{10} = c\pi\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{c\pi}{a} = \frac{3*10^8*\pi}{2.28*10^{-2}} = 4.13*10^{10}$$

$$\nu_{10} = \frac{\omega_{10}}{2*\pi} = \frac{4.13*10^{10}}{2*\pi} = 0.66*10^{10}Hz$$ **(allowed)**

$$\nu_{20} = \frac{\omega_{20}}{2*\pi} = \frac{2c\pi/a}{2*\pi} = \frac{c}{a} = \frac{3*10^8}{2.28*10^{-2}} = 1.316*10^{10}Hz$$ **(allowed)**

$$\nu_{30} = \frac{\omega_{30}}{2*\pi} = \frac{3c\pi/a}{2*\pi} = \frac{3c}{2a} = \frac{3*3*10^8}{2*2.28*10^{-2}} = 1.97*10^{10}Hz$$ (Not allowed)

$$\nu_{01} = \frac{\omega_{01}}{2*\pi} = \frac{c\pi/b}{2*\pi} = \frac{c}{2b} = \frac{3*10^8}{2*1.01*10^{-2}} = 1.49*10^{10}Hz$$ **(allowed)**

$$\nu_{02} = \frac{\omega_{02}}{2*\pi} = \frac{2c\pi/b}{2*\pi} = \frac{c}{b} = \frac{3*10^8}{1.01*10^{-2}} = 2.97*10^{10}Hz$$ (Not allowed)

$$\nu_{11} = \frac{\omega_{01}}{2*\pi} = \frac{c\pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}{2*\pi} = \frac{c\sqrt{a^2+b^2}}{2ab} = \frac{3*10^8\sqrt{(2.28)^2+(1.01)^2}*10^{-2}}{2*2.28*1.01*10^{-4}} = 1.62*10^{10}Hz$$ **(allowed)**

(c) To excite only one mode, the frequency should be between 0.66*10[10] Hz – 1.32*10[10] Hz.

$$\lambda = \frac{c}{\nu} \rightarrow 2.28cm - 4.55\ cm$$