Chapter 10 – Potentials and Fields

10.1 The Potential Formulation

10.1.1 Scalar and Vector Potentials

In this chapter we will learn how the charge and current sources (ρ and $\vec{J}$) generate electric and magnetic fields. In other words, we see general solution to Maxwell's equations:

$$\left.\begin{array}{llll}\text{(i)} & \vec{\nabla}.\vec{\mathrm{E}} = \frac{1}{\epsilon_o}\rho & ; \text{(iii)} & \vec{\nabla}\times\vec{\mathrm{E}} = -\frac{\partial \vec{B}}{\partial t} \\ \text{(ii)} & \vec{\nabla}.\vec{\mathrm{B}} = 0 & ; \text{(iv)} & \vec{\nabla}\times\vec{\mathrm{B}} = \mu_o\vec{J} + \mu_o\epsilon_o\frac{\partial \vec{E}}{\partial t}\end{array}\right\} \quad (1)$$

Given $\rho(\vec{r},t)$ and $\vec{J}(\vec{r},t)$, the electric and magnetic fields can be determined using Coulomb's law and Biot-savart law in the static case. But we seek a general solution that will be good for time-dependent configuration as well.

This can be achieved by representing fields in terms of potentials.

In electrostatics $\vec{\nabla}\times\vec{\mathrm{E}} = 0$ and hence we can write $\vec{\mathrm{E}}$ as the gradient of a scalar potential:

$$\vec{\mathrm{E}} = \vec{\nabla}V$$

But in electrodynamics this is not possible for $\vec{\mathrm{E}}$ but the magnetic field $\vec{\mathrm{B}}$ remains divergenceless, so as in magnetostatics we can still write:

$$\vec{\mathrm{B}} = \vec{\nabla}\times\vec{\mathrm{A}} \quad \dots (2)$$

We can write Maxwell's equation (iii) as:

$$\vec{\nabla}\times\vec{\mathrm{E}} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\mathrm{A}}) = -\vec{\nabla}\times\frac{\partial\vec{\mathrm{A}}}{\partial t}$$

$$\vec{\nabla}\times\left(\vec{\mathrm{E}} + \frac{\partial\vec{\mathrm{A}}}{\partial t}\right) = 0$$

The curl of $\left(\vec{\mathrm{E}} + \frac{\partial\vec{\mathrm{A}}}{\partial t}\right)$ vanishes, so we can write this quantity as a gradient of a scalar:

$$\vec{\mathrm{E}} + \frac{\partial\vec{\mathrm{A}}}{\partial t} = -\vec{\nabla}V$$

$$\vec{\mathrm{E}} = -\vec{\nabla}V - \frac{\partial\vec{\mathrm{A}}}{\partial t} \quad \dots (3)$$

When $\vec{\mathrm{A}}$ will be constant then $\vec{\mathrm{E}} = -\vec{\nabla}V$ like before.

Now using equation (3) into Maxwell's equation (i), we get:

$$\vec{\nabla}.\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right) = \frac{1}{\epsilon_o}\rho$$

$$\nabla^2 V + \frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{A}\right) = -\frac{1}{\epsilon_o}\rho \quad \dots (4)$$

In the static case, this equation will reduce to Poisson equation.

Now using equations (2) and (3) in Maxwell's equation (iv), we get

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \mu_o \vec{J} + \mu_o \epsilon_o \frac{\partial}{\partial t}\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2 \vec{A}$$

$$\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right) - \nabla^2 \vec{A} = \mu_o \vec{J} - \mu_o \epsilon_o \vec{\nabla}\frac{\partial V}{\partial t} - \mu_o \epsilon_o \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\left(\nabla^2 \vec{A} - \mu_o \epsilon_o \frac{\partial^2 \vec{A}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}.\vec{A} + \mu_o \epsilon_o \frac{\partial V}{\partial t}\right) = -\mu_o \vec{J} \quad \dots (5)$$

Equations (4) and (5) contain all the information in Maxwell's equation.

Example 1:

Find the charge and current distribution that would give rise to the potentials:

$$V = 0 \ , \ \ \vec{A} = \begin{cases} \frac{\mu_o k}{4c}(ct - |x|)^2 \hat{z} & for\ |x| < ct \\ 0 & for\ |x| > ct \end{cases}$$

Where k is a constant and $c = 1/\sqrt{\epsilon_o \mu_o}$

Solution:

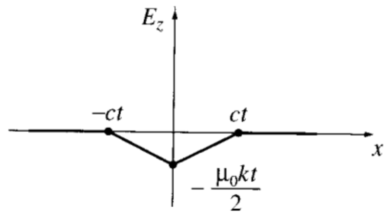
First, let's determine the electric and magnetic fields using equations (2) and (3):

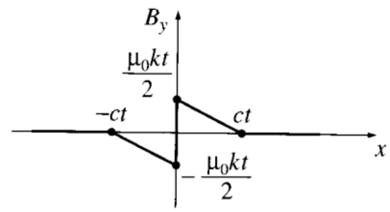
For $|x| < ct$:

$$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\mu_o k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2 \hat{y} = \pm \frac{\mu_o k}{2c}(ct - |x|)\hat{y}$$

Plus for $x > 0$ and minus for $x < 0$.

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_o k}{2c}(ct - |x|)\hat{z}$$





For $|x| > ct$:

$$\vec{\mathrm{E}} = \vec{\mathrm{B}} = 0$$

$$\vec{\nabla}.\vec{\mathrm{E}} = 0 \;\; ; \;\; \vec{\nabla}.\vec{\mathrm{B}} = 0 \;\; ;$$

$$\vec{\nabla} \times \vec{\mathrm{E}} = \mp \frac{\mu_o k}{2}\hat{y} \;\; ; \;\; \vec{\nabla} \times \vec{\mathrm{B}} = -\frac{\mu_o k}{2c}\hat{z}$$

$$\frac{\partial \vec{\mathrm{E}}}{\partial t} = -\frac{\mu_o kc}{2}\hat{z}$$

$$\frac{\partial \vec{\mathrm{B}}}{\partial t} = \pm\frac{\mu_o k}{2}\hat{y}$$

We note that Maxwell's equations are satisfied with ρ and $\vec{J}$ both equal to zero.

We also note that $\vec{B}$ has discontinuity at $x = 0$, and this indicates the presence of surface current $\vec{K}$ in the yz-plane, the boundary condition for the parallel component of the magnetic field:

$$\frac{1}{\mu_1}B_1^{\parallel} - \frac{1}{\mu_2}B_2^{\parallel} = \vec{K}_f \times \hat{n}$$

$$\frac{1}{\mu_o}\left(\frac{\mu_o kt}{2}\right) - \frac{1}{\mu_o}\left(\frac{-\mu_o kt}{2}\right) = \; kt\,\hat{y} = \vec{K} \times \hat{x}$$

Gives:

$$\vec{K} = kt\;\hat{z}$$

Evidently there is a uniform surface current in the z-direction over the plane at $x = 0$ which starts up at $t = 0$ and increases with time.

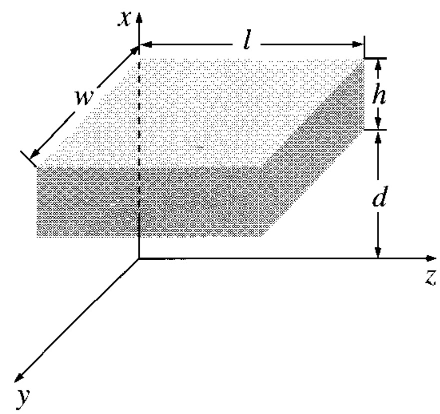
Example 2:

For the configuration in the above example 1, consider a rectangular box of length *l*, width w, and height h, situated a distance d above the yz plane.

(a) Find the energy in the box at time $t_1 = d/c$ and $t_2 = (d+h)/c$.

(b) Find the Poynting vector, and determine the energy per unit time flowing into the box during the interval $t_1 < t < t_2$.

(c) Integrate the results in part (b) from t_1 to t_2 and confirm the increase in energy (in part (a)) equals the net flux.



Solution:

(a) $$W = \frac{1}{2}\int\left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right)d\tau$$

At $t_1 = d/c,\ x \geq d = ct_1$, so $\vec{E} = 0$ and $\vec{B} = 0$ and hence $W(t_1) = 0$

At $t_2 = (d+h)/c\ ,\ ct_2 = d+h$

We found in example-1 that:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\,\hat{\mathbf{z}},$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\,\hat{\mathbf{y}} = \pm\frac{\mu_0 k}{2c}(ct - |x|)\,\hat{\mathbf{y}},$$

$$\vec{E} = -\frac{\mu_o k}{2}(d+h-x)\hat{z}\ ,\ \vec{B} = \frac{1}{c}\frac{\mu_o k}{2}(d+h-x)\hat{y}$$

Hence $B^2 = \frac{1}{c^2}E^2$ and

$$\epsilon_o E^2 + \frac{1}{\mu_o}B^2 = \epsilon_o E^2 + \frac{1}{\mu_o c^2}E^2 = 2\epsilon_o E^2 \ \ \text{[where } c^2 = 1/\epsilon_o\mu_o\text{]}$$

$$W = \frac{1}{2}\int\left(\epsilon_o E^2 + \frac{1}{\mu_o}B^2\right)d\tau = \frac{1}{2}(2\epsilon_o)\int_d^{d+h}\frac{\mu_o^2 k^2}{4}(d+h-x)^2\,dx(lw)$$

$$W = \frac{\epsilon_o\mu_o^2 k^2 lw}{4}\left.\frac{(d+h-x)^3}{-3}\right|_d^{d+h} = \frac{\epsilon_o\mu_o^2 k^2 lwh^3}{12}$$

(b) $$\vec{S}(x) = \frac{1}{\mu_o}\left(\vec{E}\times\vec{B}\right) = \frac{1}{\mu_o c}E^2[-\hat{z}\times(\pm\hat{y})] = \pm\frac{1}{\mu_o c}E^2\hat{x} = \pm\frac{\mu_o k^2}{4c}(ct-|x|)^2\hat{x}$$

For $|x| > ct$, $\vec{S} = 0$

So the energy per unit time entering the box in this time interval is:

$$\frac{dW}{dt} = P = \int \vec{S}.d\vec{a} = \frac{\mu_o k^2 lw}{4c}(ct - d)^2$$

Note that no energy flow out the top since $\vec{S}(d+h) = 0$

(c) $W = \int_{t_1}^{t_2} Pdt = \frac{\mu_o k^2 lw}{4c}\int_{\frac{d}{c}}^{\frac{d+h}{c}} (ct-d)^2 dt = \frac{\mu_o k^2 lw}{4c}\left[\frac{(ct-d)^3}{3c}\right]_{\frac{d}{c}}^{\frac{d+h}{c}} = \frac{\mu_o k^2 lwh^3}{12c^2}$

$$W = \frac{\epsilon_o \mu_o^2 k^2 lwh^3}{12}$$

Which agrees with (a).

10.1.2 Gauge Transformations

Suppose there are two sets of potentials $(V, \vec{A})$ and $(V', \vec{A}')$ which corresponds to the same electric and magnetic fields. Let's say:

$$\vec{A}' = \vec{A} + \vec{\alpha} \quad \text{and} \quad V' = V + \beta$$

Since the two $\vec{A}$'s give the same $\vec{B}$, so their curls must be equal:

$$\vec{\nabla} \times \vec{\alpha} = 0$$

We can therefore write $\vec{\alpha}$ as gradient of a scalar potential:

$$\vec{\alpha} = \vec{\nabla}\lambda$$

The two potentials also give the same $\vec{E}$, so $\left[\vec{\mathrm{E}} = -\vec{\nabla}V - \frac{\partial \vec{\mathrm{A}}}{\partial t}\right]$

$$\vec{\nabla}\beta + \frac{\partial \vec{\alpha}}{\partial t} = 0$$

$$\vec{\nabla}\left(\beta + \frac{\partial \lambda}{\partial t}\right) = 0$$

The term in parenthesis is independent of position but can depend on time, so we can write:

$$\beta = -\frac{\partial \lambda}{\partial t} + k(t)$$

Or we can absorb $k(t)$ into λ, defining a new λ by adding $\int_0^t k(t')dt'$ to the old one. Hence:

$$\vec{A}' = \vec{A} + \vec{\nabla}\lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

Hence, for any old scalar function, we can add $\vec{\nabla}\lambda$ to $\vec{\mathrm{A}}$ provided we simultaneously subtract $\frac{\partial \lambda}{\partial t}$ from V. None of this will affect the physical quantities $\vec{E}$ and $\vec{B}$. Such change in V and A are called **gauge transformations**.

10.1.3 Coulomb Gauge and Lorentz Gauge

The Coulomb Gauge:

As in magnetostatics, in Coulomb's gauge we pick:

$$\vec{\nabla}.\vec{A} = 0$$

Using this the scalar potential equation $[\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{\mathrm{A}}) = -\frac{1}{\epsilon_o}\rho]$ becomes:

$$\nabla^2 V = -\frac{1}{\epsilon_o}\rho$$

Which is the Poisson's equation and we know the solution is:

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\vec{r}', t)}{\Delta r} d\tau'$$

But unlike electrostatics, V itself is not enough to find the electric field $\vec{E}$, we need to know $\vec{A}$ as well because $\left[\vec{\mathrm{E}} = -\vec{\nabla}V - \frac{\partial \vec{\mathrm{A}}}{\partial t}\right]$.

The scalar potential in the Coulomb gauge is determined by the charge distribution right now. But electric field will change only after sufficient time has elapsed for the "news" to arrive.

In Coulomb gauge scalar potential is easier to calculate but the vector potential is cumbersome. The differential equation for $\vec{\mathrm{A}}$ in the Coulomb gauge is:

$$\nabla^2\vec{\mathrm{A}} - \mu_o\epsilon_o\frac{\partial^2\vec{\mathrm{A}}}{\partial t^2} = -\mu_o\vec{J} + \mu_o\epsilon_o\nabla\left(\frac{\partial V}{\partial t}\right)$$

Lorentz Gauge:

In the Lorentz gauge we pick:

$$\vec{\nabla}.\vec{\mathrm{A}} = -\mu_o\epsilon_o\frac{\partial V}{\partial t}$$

This is designed to eliminate the middle term in equation(5)

$$\left(\nabla^2\vec{\mathrm{A}} - \mu_o\epsilon_o\frac{\partial^2\vec{\mathrm{A}}}{\partial t^2}\right) - \vec{\nabla}\left(\vec{\nabla}.\vec{\mathrm{A}} + \mu_o\epsilon_o\frac{\partial V}{\partial t}\right) = -\mu_o\vec{J}$$

Which becomes:

$$\nabla^2\vec{A} - \mu_o\epsilon_o\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_o\vec{J}$$

The differential equation for V is:

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla}.\vec{A}) = \nabla^2 V - \mu_o\epsilon_o\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_o}\rho$$

Where

$$\nabla^2 - \mu_o\epsilon_o\frac{\partial^2}{\partial t^2} = \boxdot^2 \quad \text{d'Alembertian}$$

$$\boxed{\begin{aligned} \boxdot^2\, V &= -\frac{1}{\epsilon_o}\rho \\ \boxdot^2\, \vec{A} &= -\mu_o\vec{J} \end{aligned}} \qquad \ldots(6)$$

This democratic treatment of both potentials is nice in the context of special relativity where d'Alembertian is the natural generalization of the Laplacian and equation (6) above is regarded as four-dimensional version of Poisson's equation.

In Lorentz gauge V and $\vec{A}$ satisfy the **inhomogeneous wave equation** with a source term (in place of zero) on the right.

10.2 Continuous Distribution:

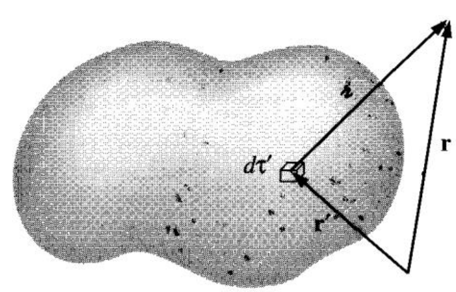
10.2.1 Retarded Potential

In the static case equation (6) $[\nabla^2 V - \mu_o\epsilon_o\frac{\partial^2 V}{\partial t^2}V = -\frac{1}{\epsilon_o}\rho$ and $\nabla^2\vec{A} - \mu_o\epsilon_o\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_o\vec{J}]$ will be reduced to the following forms:

$$\nabla^2 V = -\frac{1}{\epsilon_o}\rho \quad ; \quad \nabla^2\vec{A} = -\mu_o\vec{J}$$

With the familiar solutions:

$$\left.\begin{aligned} V(\vec{\boldsymbol{r}}) &= \frac{1}{4\pi\epsilon_o}\int\frac{\rho(\vec{r}\,')}{\Delta r}d\tau' \\ \vec{A}(\vec{\boldsymbol{r}}) &= \frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}\,')}{\Delta r}d\tau' \end{aligned}\right\} \quad \ldots(7)$$



Where Δr is the distance from the source point $\vec{r}'$ to the field point $\vec{r}$. The electromagnetic "news" travels at the speed of light. In the non-static case, therefore it is not the status of the source right now that matters but rather its condition at some earlier time t_r (called the **retarded time**), when the "message" left. Since this message must travel a distance Δr , the delay is $\Delta r/c$.

$$t_r = t - \frac{\Delta r}{c}$$

The natural generalization of equation (7) is:

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\vec{r}', t_r)}{\Delta r} d\tau'$$

$$\vec{A}(\vec{\boldsymbol{r}}, t) = \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r)}{\Delta r} d\tau'$$

Here $\rho(\vec{r}', t_r)$ is the charge density that prevailed at point $\vec{r}'$ at the retarded time t_r. Because the integrals are evaluated at the retarded time, these are called **retarded potentials**.

The most distant parts of the charge distribution have earlier retarded times than nearby ones. Like the light from the stars, what we see now left the stars at the retarded time corresponding to the star's distance from the earth.

And the retarded potentials reduced to equation (7) in the static case.

By the way, the same argument can not be applied to the fields:

$$E(\vec{\boldsymbol{r}}, t) \neq \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\vec{r}', t_r)}{\Delta r^2}(\widehat{\Delta r}) d\tau'$$

$$\vec{B}(\vec{\boldsymbol{r}}, t) \neq \frac{\mu_o}{4\pi}\int \frac{\vec{J}(\vec{r}', t_r) \times \widehat{\Delta r}}{\Delta r^2} d\tau'$$

To prove if the retarded potentials are in fact correct we need to see if they satisfy the **inhomogeneous wave equation**.

$$\nabla^2 V = \mu_o\epsilon_o\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_o}\rho$$

Lets' calculate the left-hand side of the inhomogeneous equation,

$$\vec{\nabla}V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\left[(\vec{\nabla}\rho)\frac{1}{\Delta r}+\rho\vec{\nabla}\left(\frac{1}{\Delta r}\right)\right]d\tau'$$

$$\vec{\nabla}\rho=\dot{\rho}\vec{\nabla}t_r=-\frac{1}{c}\dot{\rho}\vec{\nabla}(\Delta r)$$

$$\vec{\nabla}(\Delta r)=\widehat{\Delta r}\ \text{ and }\ \vec{\nabla}\left(\frac{1}{\Delta r}\right)=-\frac{\widehat{\Delta r}}{(\Delta r)^2}$$

$$\vec{\nabla}V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\left[-\frac{\dot{\rho}}{c}\frac{\widehat{\Delta r}}{\Delta r}-\rho\frac{\widehat{\Delta r}}{(\Delta r)^2}\right]d\tau'$$

Now taking the divergence of the above equation, we get:

$$\nabla^2V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\left[-\frac{1}{c}\left(\vec{\nabla}\dot{\rho}.\frac{\widehat{\Delta r}}{\Delta r}+\dot{\rho}\vec{\nabla}.\frac{\widehat{\Delta r}}{\Delta r}\right)-\left(\vec{\nabla}\rho.\frac{\widehat{\Delta r}}{(\Delta r)^2}+\rho\vec{\nabla}.\frac{\widehat{\Delta r}}{(\Delta r)^2}\right)\right]d\tau'$$

$$\nabla^2V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\left[-\frac{1}{c}\left(\vec{\nabla}\dot{\rho}.\frac{\widehat{\Delta r}}{\Delta r}+\dot{\rho}\vec{\nabla}.\frac{\widehat{\Delta r}}{\Delta r}\right)-\left(-\frac{1}{c}\frac{\dot{\rho}}{(\Delta r)^2}+\rho\vec{\nabla}.\frac{\widehat{\Delta r}}{(\Delta r)^2}\right)\right]d\tau'$$

$$\vec{\nabla}\dot{\rho}=-\frac{1}{c}\ddot{\rho}\vec{\nabla}(\Delta r)=-\frac{1}{c}\ddot{\rho}\widehat{\Delta r}$$

$$\vec{\nabla}.\left(\frac{\widehat{\Delta r}}{\Delta r}\right)=\frac{1}{(\Delta r)^2}$$

$$\vec{\nabla}.\left(\frac{\widehat{\Delta r}}{(\Delta r)^2}\right)=4\pi\delta^3(\Delta r)$$

$$\nabla^2V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\Delta r}-\frac{1}{c}\frac{\dot{\rho}}{(\Delta r)^2}+\frac{1}{c}\frac{\dot{\rho}}{(\Delta r)^2}-\rho4\pi\delta^3(\Delta r)\right]d\tau'$$

$$\nabla^2V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\Delta r}-\rho4\pi\delta^3(\Delta r)\right]d\tau'=\frac{1}{c^2}\frac{\partial^2V}{\partial t^2}-\frac{1}{\epsilon_o}\rho(\vec{r},t)$$

This satisfies the inhomogeneous wave equation.

This proof applied equally well to the **advanced potentials**.

$$V_a(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\frac{\rho(\vec{r}',t_a)}{\Delta r}d\tau'$$

$$\vec{A}_a(\vec{\boldsymbol{r}}, t) = \frac{\mu_o}{4\pi} \int \frac{\vec{J}(\vec{r}', t_a)}{\Delta r} d\tau'$$

Where $t_a = t + \frac{\Delta r}{c}$

Although the advanced potentials are entirely consistent with the Maxwell's equations, they **violate the principle of Causality**. They suggest that potential now depend on the charge and current distribution in the future. Here the effect precedes the cause, therefore the advanced potentials have no physical significance.

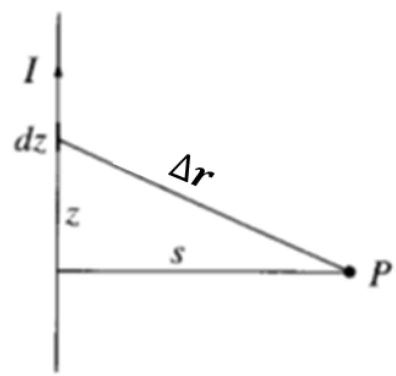
Example 3:

An infinite straight wire carries the current:

$$I(t) = \begin{cases} 0 \ \ for\ t \leq 0 \\ I_o\ for\ t > 0 \end{cases}$$

A constant current I_o is turned ON abruptly at $t = 0$. Find the resulting electric and magnetic fields.

Solution:



Assuming that wire is electrically neutral, meaning there is no extra charge accumulation on the wire, hence $V = 0$.

If we consider the wire to be oriented along z-axis and the retarded potential at point P would be:

$$\vec{A}(s, t) = \frac{\mu_o}{4\pi} \int_{-\infty}^{+\infty} \frac{I(t_r)}{\Delta r} dz\, \hat{z}$$

For $t < s/c$ the "news" has not yet arrived at P and so the vector potential is zero.

For $t > s/c$ only the segment:

$|z| \leq \sqrt{(ct)^2 - s^2}$ contributes and outside this range $t_r = t - \frac{\Delta r}{c}$ is negative, so $I(t_r) = 0$.

$$\vec{A}(s,t) = \frac{\mu_o}{4\pi} 2 \int_0^{\sqrt{(ct)^2 - s^2}} \frac{I_o}{\sqrt{s^2 + z^2}} dz\, \hat{z}$$

$$\vec{A}(s,t) = \frac{\mu_o I_o}{2\pi} \ln\left(\sqrt{s^2 + z^2} + z\right)\Big|_0^{\sqrt{(ct)^2 - s^2}} \hat{z} = \frac{\mu_o I_o}{2\pi} \ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right) \hat{z}$$

The electric field is:

$$\vec{E}(s,t) = -\frac{\partial \vec{A}}{\partial t} = -\frac{\mu_o I_o c}{2\pi\sqrt{(ct)^2 - s^2}} \hat{z}$$

$$\vec{B}(s,t) = \vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial s} \hat{\phi} = \frac{\mu_o I_o}{2\pi s} \frac{ct}{\sqrt{(ct)^2 - s^2}} \hat{\phi}$$

After a very long time, $t \to \infty$, we get electric and magnetic fields in the static case:

$$\vec{E} = 0 \quad \text{and} \quad \vec{B} = \frac{\mu_o I_o}{2\pi s} \hat{\phi}$$

10.2.2 Jefemenko's Equation

The retarded potentials are given as:

$$V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\frac{\rho(\vec{r}',t_r)}{\Delta r}d\tau' \;\; ; \;\; \vec{A}(\vec{\boldsymbol{r}},t)=\frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}',t_r)}{\Delta r}d\tau'$$

Electric and magnetic fields can be determined using:

$$\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t} \text{ and } \vec{B}=\vec{\nabla}\times\vec{A}$$

We have to be careful because the integrand of potentials depend on r through $\overrightarrow{\Delta r}=\vec{r}-\vec{r}'$ and $t_r=t-\Delta r/c$.

We have calculated $\vec{\nabla}V$:

$$\vec{\nabla}V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\int\left[-\frac{\dot{\rho}}{c}\frac{\widehat{\Delta r}}{\Delta r}-\rho\frac{\widehat{\Delta r}}{(\Delta r)^2}\right]d\tau'$$

$$\frac{\partial\vec{A}}{\partial t}=\frac{\mu_o}{4\pi}\int\frac{\partial\vec{J}/\partial t}{\Delta r}d\tau'=\frac{\mu_o}{4\pi}\int\frac{\dot{\boldsymbol{J}}}{\Delta r}d\tau'$$

Hence:

$$\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}=-\frac{1}{4\pi\epsilon_o}\int\left[-\frac{\dot{\rho}}{c}\frac{\widehat{\Delta r}}{\Delta r}-\rho\frac{\widehat{\Delta r}}{(\Delta r)^2}\right]d\tau'-\frac{\mu_o}{4\pi}\int\frac{\dot{\boldsymbol{J}}}{\Delta r}d\tau'$$

$$\vec{E}=\frac{1}{4\pi\epsilon_o}\int\left[\frac{\dot{\rho}}{c}\frac{\widehat{\Delta r}}{\Delta r}+\rho\frac{\widehat{\Delta r}}{(\Delta r)^2}-\frac{\dot{\boldsymbol{J}}}{c^2\Delta r}\right]d\tau'$$

$$\vec{B}=\vec{\nabla}\times\vec{A}=\frac{\mu_o}{4\pi}\int\left[\frac{\vec{\nabla}\times\vec{J}}{\Delta r}+\vec{J}\times\vec{\nabla}\left(\frac{1}{\Delta r}\right)\right]d\tau'$$

$$\left(\vec{\nabla}\times\vec{J}\right)_x=\frac{\partial J_z}{\partial y}-\frac{\partial J_y}{\partial z}=\dot{J}_z\frac{\partial t_r}{\partial y}-\dot{J}_y\frac{\partial t_r}{\partial z}=-\frac{1}{c}\dot{J}_z\frac{\partial(\Delta r)}{\partial y}+\frac{1}{c}\dot{J}_y\frac{\partial(\Delta r)}{\partial z}$$

$$\left(\vec{\nabla}\times\vec{J}\right)_x=\frac{1}{c}\left[\dot{\boldsymbol{J}}\times(\vec{\nabla}r)\right]_x$$

$$\vec{\nabla}\times\vec{J}=\frac{1}{c}\dot{\boldsymbol{J}}\times\vec{\nabla}r \;\; \text{and} \;\; \vec{\nabla}(1/\Delta r)=-\widehat{\Delta r}/(\Delta r)^2$$

Hence $$\vec{B}=\frac{\mu_o}{4\pi}\int\left[\frac{\dot{J}(\vec{r}',t_r)}{c\Delta r}+\frac{\vec{J}(\vec{r}',t_r)}{(\Delta r)^2}\right]\times\widehat{\Delta r}\,d\tau'$$

This is the time dependent generalization of Biot-Savart law to which it reduces in the static case.

These equations for $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ are called Jefimenko's equations. But it is easier to calcualte retarded potentials than calcualting retarded field.

10.3 Point Charges

10.3.1 Liénard-Wiechert Potentials

Lets calculate the retarded potential due to a point charge that is moving on a specific trajectory.

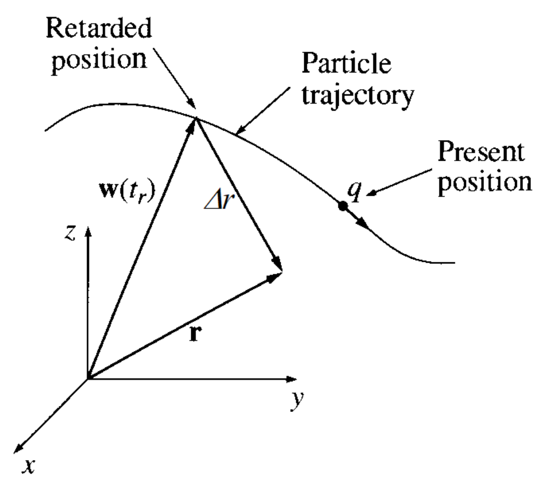
Let W(t) be the position of charge q at time t,

The retarded time is determined as:

$$t_r = t - \frac{|\vec{r} - \vec{w}(t_r)|}{c}$$

Here $|\vec{r} - \vec{w}(t_r)|$ is the distance the "news" must travel and $(t - t_r)$ is the time it takes to make the trip. Where $w(t_r)$ is the retarded position of the charge and $\Delta\vec{r}$ is the vector from the **retarded position** to the field point $\vec{r}$.

$$\Delta\vec{r} = \vec{r} - \vec{w}(t_r)$$



It is important to note that **at most one point** on the trajectory is in "communication" with $\vec{r}$ at any particular time t.

Let's assume there are two such points with retarded times t_1 and t_2

$$\Delta\vec{r}_1 = c(t - t_1) \quad \text{and} \quad \Delta\vec{r}_2 = c(t - t_2)$$

$$\Delta\vec{r}_1 - \Delta\vec{r}_2 = c(t_2 - t_1)$$

So the average velocity of the particle in the direction of $\vec{r}$ would have to be c (the velocity of light) and it might have velocity in the other directions as well, and we know that particle does not move with the velocity of light and hence it suggests that **only one retarded point contributes to the potentials at any given moment**.

$$V(\vec{\boldsymbol{r}},t) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\vec{r}',t_r)}{\Delta r}d\tau' \neq \frac{1}{4\pi\epsilon_o}\frac{q}{\Delta r}$$

For a point source Δr comes out of the integral but $\int \rho(\vec{r}',t_r)d\tau' \neq q$.

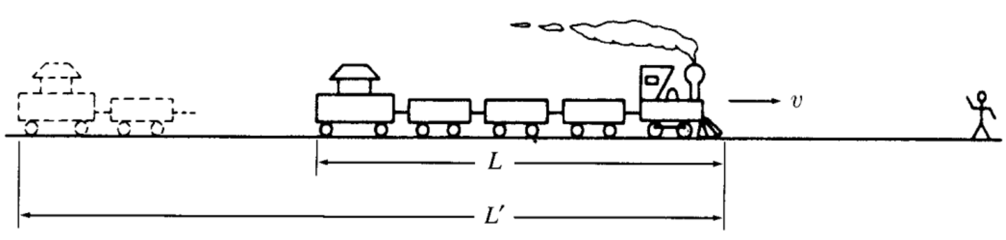
For the static case the total charge is achieved by integrating ρ over the entire charge distribution at one instant.

But for the moving charge we need to evaluate ρ at different times for different parts of the configuration. If the source is moving this gives us the distorted picture of the total charge.

In Maxwell's electrodynamics, a point charge must be regarded as the limit of an extended charge, when the size goes to zero. And for an extended particle, no matter how small:

$$\int \rho(\vec{r}',t_r)d\tau' = \frac{q}{1-\widehat{\Delta r}.\frac{\vec{v}}{c}}$$

This is a purely geometrical effect. For example, a train coming towards observer appears longer than a static train and similarly train moving away from the observer appears shorter than its static length. Because the light received by the observer from the end of the train left earlier than the light from the front of the train.



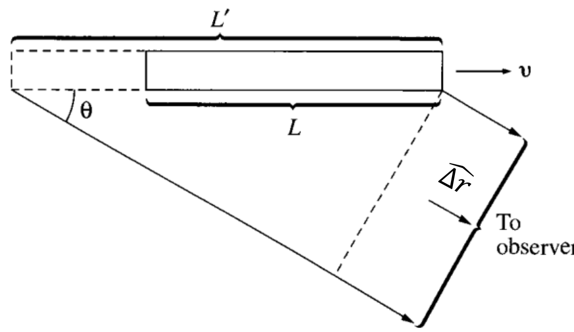
In the interval it takes light from the end of the train to travel extra distance L', the train itself moves a distance $L' - L$:

$$\frac{L'}{c} = \frac{L'-L}{v} \quad \text{or} \quad L' = \frac{L}{1-v/c}$$

So approaching train appear longer by a factor of $(1 - v/c)^{-1}$. And when a train is moving away from an observer it appears shorter by a factor of $(1 + v/c)^{-1}$. And if the train's velocity makes an angle θ with the line of sight of the observer, the extra distance light travels from the end of the train is $L' \cos\theta$, In the time $\left(\frac{L'\cos\theta}{c}\right)$ the train moves a distance of $(L' - L)$:

$$\frac{L'\cos\theta}{c} = \frac{L' - L}{v}$$

$$L' = \frac{L}{1 - v\cos\theta/c}$$



This effect does not distort the dimensions perpendicular to the motion. And the apparent volume of the train τ' then is related to the actual volume of the train τ :

$$\tau' = \frac{\tau}{1 - \widehat{\Delta r}.\vec{v}/c}$$

Where $\widehat{\Delta r}$ is the unit vector from the train to the observer.

Using this understanding, we can write the electric potential of a moving point charge as follows:

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_o}\int\frac{\rho(\vec{r}', t_r)}{\Delta r}d\tau' = \frac{1}{4\pi\epsilon_o}\frac{q}{\Delta r\left(1 - \widehat{\Delta r}.\vec{v}/c\right)}$$

$$V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_o}\frac{qc}{\left(\Delta rc - \overrightarrow{\Delta r}.\vec{v}\right)}$$

Where $\vec{v}$ is the velocity of the charge at the retarded time, and $\overrightarrow{\Delta r}$ is the vector from the retarded position to the field point $\vec{r}$.

Since the current density of a rigid body is $\rho\vec{v}$, so we can write the vector potential as:

$$\vec{A}(\vec{\boldsymbol{r}}, t) = \frac{\mu_o}{4\pi}\int\frac{\vec{J}(\vec{r}', t_r)}{\Delta r}d\tau' = \frac{\mu_o}{4\pi}\frac{\vec{v}}{\Delta r}\int\rho(r', t_r)d\tau'$$

$$\vec{A}(\vec{\boldsymbol{r}},t)=\frac{\mu_o}{4\pi}\frac{qc\vec{v}}{\left(\Delta rc-\overrightarrow{\Delta r}.\vec{v}\right)}=\frac{\vec{v}}{c^2}V(\vec{r},t)$$

10.3.2 The Fields of a Moving Point Charge

Scalar and vector potentials due to a moving charge are given as:

$$V(\vec{\boldsymbol{r}},t)=\frac{1}{4\pi\epsilon_o}\frac{qc}{(\Delta rc-\overrightarrow{\Delta r}.\vec{v})} \text{ and } \vec{A}(\vec{\boldsymbol{r}},t)=\frac{\vec{v}}{c^2}V(\vec{r},t)$$

Electric and magnetic fields can be found as:

$$\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t} \quad \text{and} \quad \vec{B}=\vec{\nabla}\times\vec{A}$$

Note that: $\overrightarrow{\Delta r}=\vec{r}-\vec{w}(t_r)$ and $\vec{v}=\dot{w}(t_r)$, both are evaluated at the retarded time and t_r defined by the equation: $|\vec{r}-\vec{w}(t_r)|=c(t-t_r)$ is iteself a function of $\vec{r}$ and t.

$$\vec{\nabla}V=\frac{qc}{4\pi\epsilon_o}\frac{-1}{\left(\Delta rc-\overrightarrow{\Delta r}.\vec{v}\right)^2}\vec{\nabla}\left(\Delta rc-\overrightarrow{\Delta r}.\vec{v}\right)$$

Since $\overrightarrow{\Delta r}=c(t-t_r)$ so $\vec{\nabla}(\Delta r)=-c\vec{\nabla}t_r$

As for the second term, product rule is:

$$\vec{\nabla}\left(\overrightarrow{\Delta r}.\vec{v}\right)=\left(\overrightarrow{\Delta r}.\vec{\nabla}\right)\vec{v}+\left(\vec{v}.\vec{\nabla}\right)\overrightarrow{\Delta r}+\overrightarrow{\Delta r}\times\left(\vec{\nabla}\times\vec{v}\right)+\vec{v}\times\left(\vec{\nabla}\times\overrightarrow{\Delta r}\right)$$

$$\left(\overrightarrow{\Delta r}.\vec{\nabla}\right)\vec{v}=\left(\Delta r_x\frac{\partial}{\partial x}+\Delta r_y\frac{\partial}{\partial y}+\Delta r_z\frac{\partial}{\partial z}\right)\vec{v}(t_r)$$

$$=\Delta r_x\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial x}+\Delta r_y\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial y}+\Delta r_z\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial z}$$

$$\boldsymbol{\left(\overrightarrow{\Delta r}.\vec{\nabla}\right)\vec{v}=\vec{a}\left(\overrightarrow{\Delta r}.\vec{\nabla}t_r\right)}$$

Where $\vec{a}=d\vec{v}/dt_r$ is the acceleration of the charge particle at the retarded time.

$$\boldsymbol{\left(\vec{v}.\vec{\nabla}\right)\overrightarrow{\Delta r}}=\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)\left(x\hat{\imath}+y\hat{\jmath}+z\hat{k}\right)=v_x\hat{\imath}+v_y\hat{\jmath}+v_z\hat{k}=\boldsymbol{\vec{v}}$$

$$\vec{\nabla}\times\vec{v}=\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\hat{k}$$

$$= \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\imath} + \left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\jmath} + \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{k}$$

$$\vec{\nabla} \times \vec{v} = -\vec{a} \times \vec{\nabla} t_r$$

And $$\vec{\nabla} \times \overrightarrow{\Delta r} = \vec{\nabla} \times \vec{r} - \vec{\nabla} \times \vec{w}$$

$$\vec{\nabla} \times \vec{r} = 0$$

And $$\vec{\nabla} \times \vec{w} = -\vec{v} \times \vec{\nabla} t_r$$

So $$\vec{\nabla} \times \overrightarrow{\Delta r} = \vec{v} \times \vec{\nabla} t_r$$

$$\vec{\nabla}(\overrightarrow{\Delta r}.\vec{v}) = \vec{a}(\overrightarrow{\Delta r}.\vec{\nabla} t_r) + \vec{v} - \overrightarrow{\Delta r} \times (\vec{a} \times \vec{\nabla} t_r) + \vec{v} \times (\vec{v} \times \vec{\nabla} t_r)$$

$$\vec{\nabla}(\overrightarrow{\Delta r}.\vec{v}) = \vec{v} + (\overrightarrow{\Delta r}.\vec{a} - v^2)\vec{\nabla} t_r$$

So our equation for gradient V, becomes:

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_o}\frac{-1}{\left(\Delta rc - \overrightarrow{\Delta r}.\vec{v}\right)^2}\vec{\nabla}(\Delta rc - \overrightarrow{\Delta r}.\vec{v})$$

$$\vec{\nabla} V = \frac{qc}{4\pi\epsilon_o}\frac{-1}{\left(\Delta rc - \overrightarrow{\Delta r}.\vec{v}\right)^2}\left[v + (c^2 - v^2 + \overrightarrow{\Delta r}.\vec{a})\vec{\nabla} t_r\right]$$

Where $\vec{\nabla} t_r = \vec{\nabla}\left(t - \frac{\Delta r}{c}\right) = -\frac{1}{c}\vec{\nabla}(\Delta r)$

$$-c\vec{\nabla} t_r = \vec{\nabla}(\Delta r) = \vec{\nabla}\sqrt{\overrightarrow{\Delta r}.\overrightarrow{\Delta r}} = \frac{1}{2\sqrt{\overrightarrow{\Delta r}.\overrightarrow{\Delta r}}}\vec{\nabla}(\overrightarrow{\Delta r}.\overrightarrow{\Delta r})$$

$$= \frac{1}{\mathscr{r}}[(\boldsymbol{\mathscr{r}} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \boldsymbol{\mathscr{r}})].$$

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$$\nabla V = \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^3}\left[(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})\mathbf{v} - (c^2 - v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a})\boldsymbol{\mathscr{r}}\right].$$

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &= \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\Big[(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})(-\mathbf{v} + \mathscr{r}\mathbf{a}/c) \\ &+ \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\Big].\end{aligned}$$

Combining these results, and introducing the vector

$$\mathbf{u} \equiv c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v},$$

I find

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})].}$$

Meanwhile,

$$\nabla\times\mathbf{A} = \frac{1}{c^2}\nabla\times(V\mathbf{v}) = \frac{1}{c^2}[V(\nabla\times\mathbf{v}) - \mathbf{v}\times(\nabla V)].$$

We have already calculated $\nabla\times\mathbf{v}$ (Eq. 10.55) and ∇V (Eq. 10.62). Putting these together,

$$\nabla\times\mathbf{A} = -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathbf{u}\cdot\boldsymbol{\mathscr{r}})^3}\boldsymbol{\mathscr{r}}\times[(c^2 - v^2)\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}].$$

The quantity in brackets is strikingly similar to the one in Eq. 10.65, which can be written, using the BAC-CAB rule, as $[(c^2 - v^2)\mathbf{u} + (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a}]$; the main difference is that we have $\mathbf{v}$'s instead of $\mathbf{u}$'s in the first two terms. In fact, since it's all crossed into $\boldsymbol{\mathscr{r}}$ *anyway*, we can with impunity *change* these $\mathbf{v}$'s into $-\mathbf{u}$'s; the extra term proportional to $\hat{\boldsymbol{\mathscr{r}}}$ disappears in the cross product. It follows that

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E}(\mathbf{r}, t).}$$

Evidently *the magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.*

The first term in $\mathbf{E}$ (the one involving $(c^2 - v^2)\mathbf{u}$) falls off as the inverse *square* of the distance from the particle. If the velocity and acceleration are both zero, this term alone survives and reduces to the old electrostatic result

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}.$$

For this reason, the first term in **E** is sometimes called the **generalized Coulomb field**. (Because it does not depend on the acceleration, it is also known as the **velocity field**.) The second term (the one involving $\hat{\boldsymbol{\mathscr{r}}} \times (\mathbf{u} \times \mathbf{a})$) falls off as the inverse *first* power of $\mathscr{r}$ and is therefore dominant at large distances. As we shall see in Chapter 11, it is this term that is responsible for electromagnetic radiation; accordingly, it is called the **radiation field**—or, since it is proportional to a, the **acceleration field**. The same terminology applies to the magnetic field.

Example 10.4

Calculate the electric and magnetic fields of a point charge moving with constant velocity.

Solution: Putting $\mathbf{a} = 0$ in Eq. 10.65,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{(c^2 - v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3} \mathbf{u}.$$

In this case, using $\mathbf{w} = \mathbf{v}t$,

$$\mathscr{r}\mathbf{u} = c\boldsymbol{\mathscr{r}} - \mathscr{r}\mathbf{v} = c(\mathbf{r} - \mathbf{v}t_r) - c(t - t_r)\mathbf{v} = c(\mathbf{r} - \mathbf{v}t).$$

In Ex. 10.3 we found that

$$\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v} = \boldsymbol{\mathscr{r}} \cdot \mathbf{u} = \sqrt{(c^2 t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}.$$

In Prob. 10.14, you showed that this radical could be written as

$$Rc\sqrt{1 - v^2 \sin^2\theta / c^2},$$

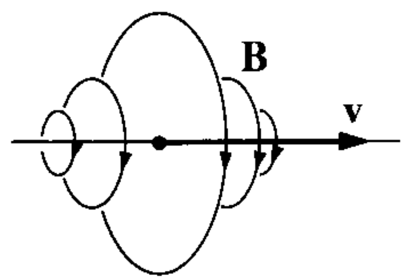
where

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$$

is the vector from the *present* location of the particle to $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Thus

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta/c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}.} \tag{10.68}$$

Notice that **E** points along the line from the *present* position of the particle. This is an *extraordinary* coincidence, since the "message" came from the *retarded* position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions **E** is *reduced* by a factor $(1 - v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1 - v^2/c^2}$.



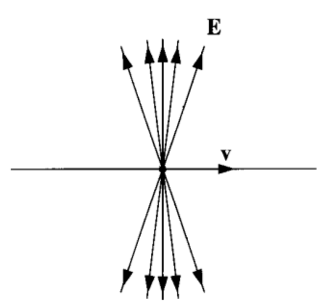
As for **B**, we have

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{\mathscr{r}} = \frac{(\mathbf{r} - \mathbf{v}t) + (t - t_r)\mathbf{v}}{\mathscr{r}} = \frac{\mathbf{R}}{\mathscr{r}} + \frac{\mathbf{v}}{c},$$

and therefore

$$\boxed{\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).} \tag{10.69}$$

Lines of **B** *circle around* the charge, as shown in Fig. 10.11.



The fields of a point charge moving at constant velocity (Eqs. 10.68 and 10.69) were first obtained by Oliver Heaviside in 1888.[13] When $v^2 \ll c^2$ they reduce to

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{\mathbf{R}}; \qquad \mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{q}{R^2} (\mathbf{v} \times \hat{\mathbf{R}}). \tag{10.70}$$

The first is essentially Coulomb's law, and the latter is the "Biot-Savart law for a point charge" I warned you about in Chapter 5 (Eq. 5.40).