Chap12-Electrodynamics and Relativity

12.1 The Special Theory of Relativity

12.1.1 Einstein's Postulates

Classical mechanics obeys the principle of relativity, the same laws apply in any **inertial reference frame**. An inertial frame is the one that is at rest or moving with a constant velocity.

In classical mechanics the principle of relativity was first given by Galileo. Does it apply in electrodynamics, the answer seems to be NO. because moving charges produce magnetic field but stationary charges don't. If we have charge glued on a train and set in motion then for an observer on the ground it produces a magnetic field but not for the observer riding the same train. May of the electrodynamics equations make reference to the velocity of the charge. This means that electromagnetic theory presupposes the existence of a unique stationary reference frame, with respect to which all the velocities are measured.

Suppose we mount a wire loop on a freight car and have the train pass between the poles of a giant magnet as shown in Fig.1 . As the loop rides through the magnetic field, a motional EMF is established:

$$\varepsilon = -\frac{d\Phi}{dt}$$

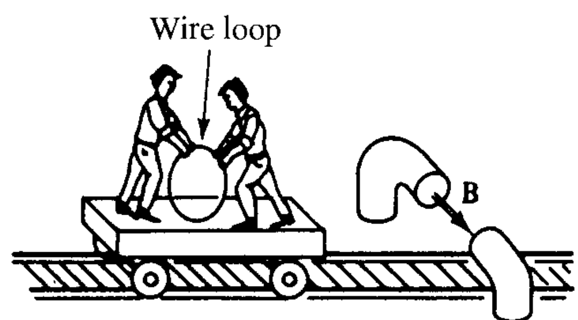


Figure 1:

This EMF is due to the magnetic force on the charges in the wire loop, which are moving along with the train.

If someone on the train applies the laws of electrodynamics in that system, he will not predict any magnetic force because the loop is not moving. However, a giant magnet will fly by, the magnetic field in the freight car will change and a changing magnetic field induces an electric field, by Faraday's law. The resulting electric field will generated an EMF in the loop:

$$\varepsilon = -\frac{d\Phi}{dt}$$

Since Faraday's law and the flux rule predicted the same EMF, observer on the train will get the right answer, *even though their physical interpretation of the situation in completely wrong*. Or Is It?

Einstein could not believe this was a mere coincidence, he instead realized that electromagnetic phenomenon, like mechanical ones, obey the **principle of relativity**.

In his view, the analysis of the observer on the train is as valid as the analysis of the observer on the ground. If their interpretations differ (one calling the process electric and the other magnetic), so be it; their actual predictions are in agreement.

Einstein explained that observable phenomenon depended on the relative motion of the conductor and the magnet. According to his predecessors the equality of the two EMF's was just a lucky coincidence, to them only one observer was right and the other was wrong.

They thought of electric and magnetic fields as strains in an invisible jelly like medium called **ETHER**, which permeated all of the space. The speed of the charge was to be measured with respect to the stationary ether- only then would the laws of electrodynamics be valid. The train observer is wrong because that frame is moving with respect to the ether.

Now among the results of classical electrodynamics is the prediction that electromagnetic waves travel through the vacuum at a speed:

$$\frac{1}{\sqrt{\epsilon_o \mu_o}} = 3.00 \times 10^8 \; m/s$$

relative to the ether. One should be able to detect the ether wind (as motorbike rider feels air wind).

Michelson and Morley performed their famous experiment and determined the following:

- The speed of light is exactly the same in all directions.

This was difficult to explain because all other waves (water waves, sound waves, etc) travel at a prescribed speed relative to the propagating medium and if this medium is in motion with respect to the observer, the net speed is always greater "downstream" than "upstream".

Einstein took the results of Michelson Morely experiment at its face value and suggested that the speed of light is a universal constant, the same in all directions, regardless of the motion of the observer or the source. ***There is no ether wind because there is no ether***.

Einstein produced his two famous postulates:

1. **The principle of relativity:** The law of physics apply in all inertial reference system.
2. **The universal speed of light:** The speed of light in vacuum is the same for all inertial observers regardless of the motion of the source.

The special theory of relativity is derived from these two postulates, the first elevates Galileo's observation about classical mechanics to the status of a general law, applying to all of physics. It states that ***there is no absolute rest system***.

The second is considered as Einstein's response to Michelson Morley experiment and it means that ***there is no ether***.

Consider a man walks 5 km/h down the corridor of a train that is moving at 60km/h, so the net speed of the person relative to the ground would be 65 km/h. The speed of person A with respect to ground C (v_{AC}) is equal to the sum of speed of the train B with respect to ground (v_{BC}) and speed of the person A with respect to train B (v_{AB}):

$$v_{AC} = v_{AB} + v_{BC} \quad \dots (1)$$

But if A is a light whether it comes from a flash light on the train, or from the lamp on the ground or from a star in the sky, according to Einstein, it will be same.

$$v_{AC} = v_{AB} = c \quad \dots (2)$$

Equation (1) is called **Galileo's velocity addition rule**, which is incompatible with the second postulate. According to **Einstein's velocity addition rule**:

$$v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB} v_{BC}/c^2)} \quad \dots (3)$$

For "ordinary" speeds ($v_{AB} \ll c$, $v_{BC} \ll c$), the denominator is so close to one that there is almost no difference in Galileo's rule and Einstein's rule. But if $v_{AB} = c$, then according to Einstein's formula:

$$v_{AC} = \frac{c + v_{BC}}{1 + (c * v_{BC}/c^2)} = c$$

Example 1:

(a) What is the percent error introduced when you use Galileo's rule, instead of Einstein's, with $v_{AB} = 5\ km/h$ and $v_{BC} = 60\ km/h$?

(b) Suppose you could run at half the speed of light down the corridor of a train going three quarters the speed of light. What would your speed be relative to the ground?

Solution:

(a)

$$v_G = v_{AB} + v_{BC} = 5 + 60 = 65\ km/h$$

$$v_E = \frac{v_{AB} + v_{BC}}{1 + (v_{AB} v_{BC}/c^2)} \approx v_G \left(1 - \frac{v_{AB} v_{BC}}{c^2}\right)$$

$$\frac{v_G - v_E}{v_G} = \frac{v_{AB} v_{BC}}{c^2} = \frac{5 * 60}{(1.08 * 10^9)^2} = 2.57 * 10^{-14}\%$$

(b)

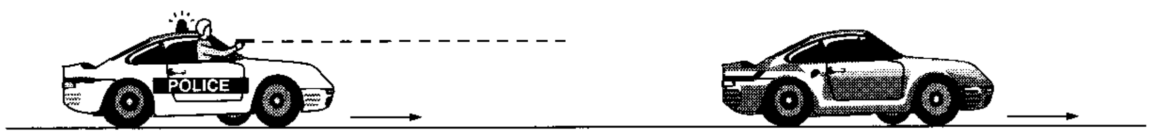
$$c = 3 * 10^8 \frac{m}{s} = 3 * 10^8 * 10^{-3} * 3600 = 1.08 * 10^9\ km/h$$

$$v_G = v_{AB} + v_{BC} = 0.5c + 0.75c = 1.25c$$

$$v_E = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)} = \frac{0.5c + 0.75c}{1 + \left(\frac{0.5c * 0.75c}{c^2}\right)} = 0.909c$$

Example 2:

As the outlaws escape in their getaway car, which goes $\frac{3}{4}c$, the police officer fires a bullet from the pursuit car, which only goes $\frac{1}{2}c$. The muzzle velocity of the bullet (relative to the gun) is $\frac{1}{3}c$. Does the bullet reach its target (a) according to Galileo, (b) according to Einstein.



(a) According to Galileo:

$$v_{BG} = v_{muzzle} + v_{police} = \frac{1}{3}c + \frac{1}{2}c = \frac{5}{6}c = \frac{10}{12}c$$

The speed of outlaws' car is $\frac{3}{4}c = \frac{9}{12}c$

This means bullet reaches the target.

(b) According to Einstein:

$$v_{BG} = \frac{v_{muzzle} + v_{police}}{1 + \left(v_{muzzle}v_{police}/c^2\right)} = \frac{\frac{1}{3}c + \frac{1}{2}c}{1 + \left(\frac{1}{6}\right)} = \frac{5}{7}c = \frac{20}{28}c$$

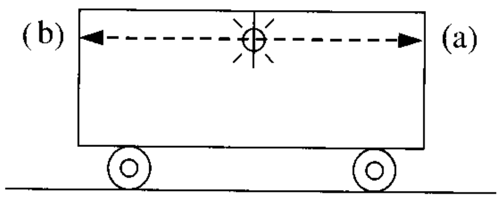
The speed of outlaws' car is $\frac{3}{4}c = \frac{21}{28}c$

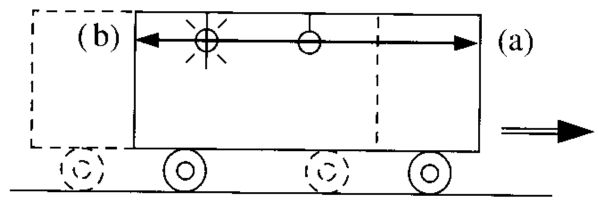
The bullet does not reach the target.

12.1.2 The Geometry of Relativity

(i) The relativity of Simultaneity

Imagine a freight car, traveling at a constant speed along a smooth track. In the very center of the car there hangs a light bulb. When someone switches it ON, the light spreads out in all directions at speed c. An ***observer on the train*** will notice that light reaches the front and back end of the car ***simultaneously***.





However, an ***observer on the ground*** will notice that these two events are ***not simultaneous***. According to him event (b) happened before event (a).

And an ***observer passing by an express train*** would report that event (a) preceded event (b).

Conclusion:

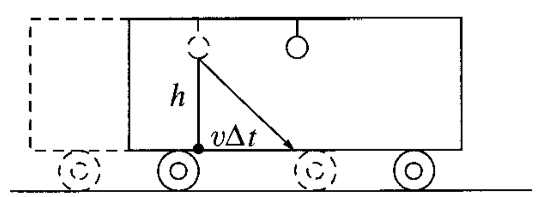
The two events that are simultaneous in one inertial system are not, in general, simultaneous in another.

(ii) Time Dilation

Now let's consider a light ray that leaves the bulb and strikes the floor of the car directly below. How long does it take for the light to make this trip?

Well, from the point of view of the observer on the train, the answer is $\Delta\bar{t} = h/c$ if the height of the bulb from the floor is h.

When observed from the ground then the ray must travel more distance because the train itself is moving and the distance travelled by the light is $\sqrt{h^2 + (v\Delta t)^2}$:



$$\Delta t = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c}$$

$$\Delta t = \frac{h}{c}\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{\Delta\bar{t}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta\bar{t} = \Delta t\sqrt{1 - v^2/c^2}$$

Hence the time elapsed between the same two events, for the two observers is different. The interval recorded on the train is shorter by a factor:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Conclusion:

Moving clocks run slow

This is called **time dilation**. Of all Einstein's prediction, time dilation was experimentally confirmed. Most elementary particles are unstable, they disintegrate after a characteristic lifetime, for example, the lifetime of neutron is 15 minutes, of a muon is 2×10^{-6}s and of a neutral pion is $9 \text{x} 10^{-17}$ s. These are lifetime at rest and when they are moving at speeds close to c then they last much longer, for their internal clocks are running slow, in accordance with Einstein's time dilation.

Example 3:

A muon is travelling through the laboratory at three-fifths the speed of light. How long does it last?

Solution:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{5}{4}$$

So it lives longer (than rest) by a factor of 5/4:

$$\frac{5}{4} * (2 \times 10^{-6}) = 2.5 \times 10^{-6} s$$

Example 4:

In a laboratory experiment a muon is observed to travel 800m before disintegrating. A graduate student looks up the lifetime of a muon ($2 \times 10^{-6}\ s$) and concludes its speed was:

$$v = \frac{800m}{2 \times 10^{-6}} = 4 \times 10^{8}\ m/s$$

Which is greater than the speed of light. Where did the student made a mistake?

Solution:

Student failed to take the time dilation into account. The lifetime of muon when moving is:

$$\tau = \gamma\bar{\tau} = \frac{\bar{\tau}}{\sqrt{1 - v^2/c^2}} = \frac{2 \times 10^{-6}}{\sqrt{1 - v^2/c^2}}$$

So the speed of muon is:

$$v = \frac{s}{\tau} = \frac{800 * \sqrt{1 - v^2/c^2}}{2 \times 10^{-6}}$$

$$v^2 = \left(\frac{800}{2 \times 10^{-6}}\right)^2 * \left(1 - \frac{v^2}{c^2}\right) = 1.6 * 10^{17} - \frac{1.6 * 10^{17}}{9 * 10^{16}} v^2 = 1.6 * 10^{17} - 1.77778v^2$$

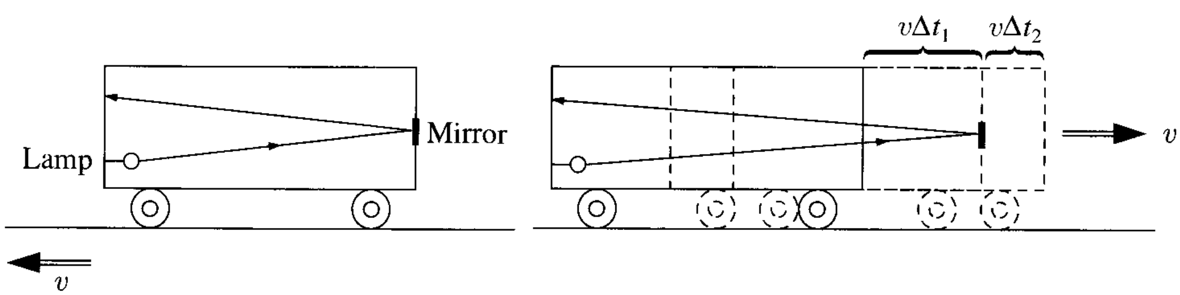
$$2.77778v^2 = 1.6 * 10^{17}$$

$$v = \sqrt{\frac{1.6 * 10^{17}}{2.77778}} = 2.4 * 10^8 \; m/s$$

Which is less than the speed of light.

(iii) Lorentz Contraction

In the third experiment, if we have a lamp at one end of the moving car and mirror at the other end. How long will it take for the light to travel to the mirror and reflect back to the lamp.



For an observer on the car:

$$\Delta\bar{t} = 2\frac{\Delta\bar{x}}{c} \quad \dots (1)$$

But for an observer on the ground, due to motion of the train, the time for the light signal to reach the front Δt_1 is different than the return time Δt_2:

$$\Delta t_1 = \frac{\Delta x + v\Delta t_1}{c}$$

$$\Delta t_2 = \frac{\Delta x - v\Delta t_2}{c}$$

Solving for Δt_1 and Δt_2:

$$\Delta t_1 = \frac{\Delta x}{c-v} \quad \text{and} \quad \Delta t_2 = \frac{\Delta x}{c+v}$$

The total time for the trip:

$$\Delta t = \Delta t_1 + \Delta t_2 = 2\frac{\Delta x}{c}\frac{1}{(1 - v^2/c^2)} \dots (2)$$

These same intervals are related by the time dilation formula, as:

$$\Delta \bar{t} = \sqrt{1 - v^2/c^2}\, \Delta t$$

Using this in equation (1) and (2), we get:

$$2\frac{\Delta \bar{x}}{c} = \sqrt{1 - v^2/c^2}\left(2\frac{\Delta x}{c}\frac{1}{(1 - v^2/c^2)}\right)$$

$$\Delta \bar{x} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\, \Delta x$$

The length of the car is not the same when measured by an observer on the ground- it is shorter by a factor $\gamma = 1/\sqrt{1 - v^2/c^2}$

Conclusion:

Moving objects are shortened

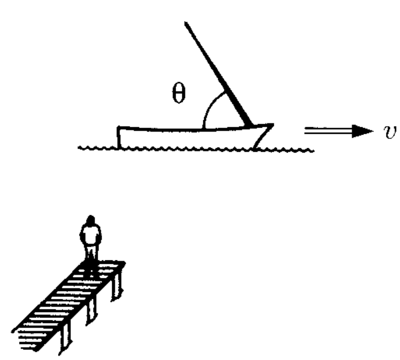
This is called ***Lorentz contraction***.

A moving object is shortened only along the direction of motion.

Dimensions perpendicular to the velocity are not contracted.

Example 5:

A sailboat is manufactured so that the mast leans at an angle $\bar{\theta}$ with respect to the deck. An observer standing on a dock sees the boat go by at speed v as shown in the figure. What angle does this observer say the mast makes?



Solution:

Let's say the length of the mass is $\bar{l}$. To an observer on the boat, height of the mast is :

$$\bar{y} = \bar{l} \sin \bar{\theta}$$

Horizontal projection of the mast is:

$$\bar{x} = \bar{l} \cos \bar{\theta}$$

To observer on the dock, the height is unaffected because there is no relative velocity in that direction, but the horizontal length is contracted:

$$x = \frac{1}{\gamma} (\bar{l} \cos \bar{\theta})$$

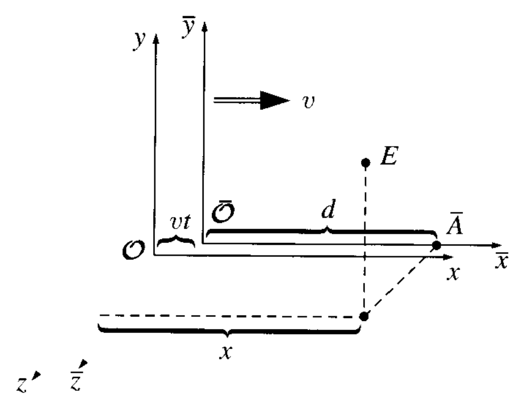
$$y = \bar{l} \sin \bar{\theta}$$

The angle seen by the observer on the dock is:

$$\tan \theta = \frac{y}{x} = \frac{\bar{l} \sin \bar{\theta}}{\frac{1}{\gamma} (\bar{l} \cos \bar{\theta})} = \gamma \tan \bar{\theta} = \frac{\tan \bar{\theta}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

12.1.3 The Lorentz Transformation

Suppose an event happens in one inertial frame S with coordinates (x, y, z, t) and we would like to translate the same event in another inertial reference frame $\bar{S}$ with coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$. We need a "dictionary" for translating for the language of S to the language of $\bar{S}$.



We may orient our axes as shown in the figure, so that $\bar{S}$ slides with speed v along the x-axis of S. If we start the clock ($t = 0$) at the moment the origins (O and $\bar{O}$) coincide. Then at time t, $\bar{O}$ will be a distance vt from O, and hence:

$$x = d + vt \ldots (1)$$

Where d is the distance from $\bar{O}$ to $\bar{A}$ at time t ($\bar{A}$ is the point on the $\bar{x}$ axis which is even with E when the event occurs.)

Before Einstein, it was believed that

$$d = \bar{x} \ldots (2)$$

And thus construct the "dictionary":

$$\left.\begin{array}{ll} \text{(i)} & \bar{x} = x - vt \\ \text{(ii)} & \bar{y} = y \\ \text{(iii)} & \bar{z} = z \\ \text{(iv)} & \bar{t} = t \end{array}\right\} \quad (3)$$

These are now called **Galilean transformations**.

In the context of special relativity, however we must expect (iv) to be replaced by a rule that incorporates dilation, the relativity of simultaneity and the non-synchronization of the moving clocks. Similarly, there will be a modification in (i) to account for the Lorentz contraction. As for (ii) and (iii), they remain unchanged because there is no change in length perpendicular to the direction of motion.

Let's examine how (i) breaks down? From equation (2), d is the distance from $\bar{O}$ to $\bar{A}$ as measured in S, whereas $\bar{x}$ is the distance from $\bar{O}$ to $\bar{A}$ as measured in $\bar{S}$. Because $\bar{O}$ and $\bar{A}$ are at rest in $\bar{S}$, so $\bar{x}$ is the "moving stick" which appears contracted to S:

$$d = \frac{1}{\gamma}\bar{x}$$

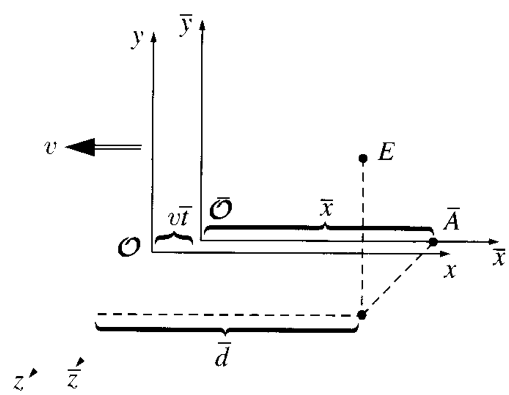
$$\bar{x} = \gamma(x - vt) \dots (4)$$

We can make the same argument form the point of view of $\bar{S}$. The figure below looks similar to the above figure, but in this case, it depicts the scene at time $\bar{t}$, whereas the above figure showed the scene at time t.

Suppose that $\bar{S}$ also starts the clock when the origins coincide, then at time $\bar{t}$, O will be a distance $v\bar{t}$ from $\bar{O}$, and therefore:

$$\bar{x} = \bar{d} - v\bar{t}$$

Where $\bar{d}$ is the distance from O to A at time $\bar{t}$, and A is that point on the x-axis which is even with E when the event occurs.



Classically $x = \bar{d}$ but according to relativity, since x is the distance from O to A in S, whereas $\bar{d}$ is the distance from O to A in $\bar{S}$. Because O and A are at rest in S , x is the "moving stick" with respect to $\bar{S}$. Hence

$$\bar{d} = \frac{1}{\gamma}x$$

$$x = \gamma(\bar{x} + v\bar{t}) \dots \ (5)$$

Using $\bar{x}$ from equation (4) and substitute in equation (5), we get:

$$\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)$$

So we get the following transformation:

$$
\left.\begin{array}{ll}
\text{(i)} & \bar{x} = \gamma(x - vt) \\
\text{(ii)} & \bar{y} = y \\
\text{(iii)} & \bar{z} = z \\
\text{(iv)} & \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right)
\end{array}\right\} \quad (6)
$$

These are the **Lorentz transformations**, with which Einstein replaced the Galilean ones. The reverse dictionary which takes $\bar{S}$ back to S, is as follows:

$$
\left.\begin{array}{ll}
\text{(i)} & x = \gamma(\bar{x} + v\bar{t}) \\
\text{(ii)} & y = \bar{y} \\
\text{(iii)} & z = \bar{z} \\
\text{(iv)} & t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right)
\end{array}\right\} \quad (7)
$$

Einstein's velocity addition rule:

Suppose a particle moves a distance dx (in S) in a time dt. Its velocity is then:

$$u = \frac{dx}{dt}$$

In $\bar{S}$, meanwhile, it moved a distance:

$$d\bar{x} = \gamma(dx - vdt)$$

And

$$d\bar{t} = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

Hence the velocity in $\bar{S}$ is:

$$\bar{u} = \frac{d\bar{x}}{d\bar{t}} = \frac{\gamma(dx - vdt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{\left(\frac{dx}{dt} - v\right)}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{u - v}{1 - uv/c^2}$$

This is **Einstein's velocity addition rule**. If we let A be the particle, B be S and C by $\bar{S}$:

then $= v_{AB}$, $\bar{u} = v_{AC}$ and $v = v_{CB} = -v_{BC}$

$$\bar{u} = v_{AC} = \frac{v_{AB} + v_{BC}}{1 + (v_{AB}v_{BC}/c^2)}$$

12.1.4 The structure of Spacetime

(i) Four Vectors

The Lorentz transformation take on a simpler appearance when expressed in terms of the quantities:

$$x^0 \equiv ct \quad , \quad \beta = v/c$$

Using x^0 (instead of t) and using β (instead of v) amount to changing the unit of time from the second to the meter- 1 meter of x^0 corresponds to the time it takes for the light to travel 1 meter (in vacuum). Similarly, we can number the x, y, and z coordinates as:

$$x^1 = x \ , \qquad x^2 = y \ , \quad x^3 = z$$

Then the Lorentz transformation is:

$$\left.\begin{array}{ll} \text{(i)} & \bar{x}^0 = \gamma(x^0 - \beta x^1) \\ \text{(ii)} & \bar{x}^1 = \gamma(x^1 - \beta x^0) \\ \text{(iii)} & \bar{x}^2 = x^2 \\ \text{(iv)} & \bar{x}^3 = x^3 \end{array}\right\} \qquad (8)$$

Or in matrix form:

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Or

$$\bar{x}^\mu = \sum_{\nu=0}^{3} (\Lambda^\mu_\nu) x^\nu$$

Where Λ is the **Lorentz transformation matrix** (the superscript μ labels the row and the subscript ν labels the column). An advantage of writing in this abstract manner is that we can handle in the same format a more general transformation where relative motion is not along a common $x\ \bar{x}$ axis; the matrix Λ will be more complicated but the structure is the same.

We can define a 4-vector as any set of four components that transform in the same manner as (x^0, x^1, x^2, x^3) under Lorentz transformations:

$$\bar{a}^{\mu} = \sum_{\nu=0}^{3} (\Lambda^{\mu}_{\nu}) a^{\nu}$$

For the particular case of the transformation along x-axis:

$$\bar{a}^0 = \gamma(a^0 - \beta a^1)$$

$$\bar{a}^1 = \gamma(a^1 - \beta a^0)$$

$$\bar{a}^2 = a^2$$

$$\bar{a}^3 = a^3$$

This is a 4-vector analog to a dot product ($\boldsymbol{A}.\boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$), but it is not just the sum of the products of the like components, rather the zeroth component have a minus sign:

$$-a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

This is the **four dimensional scalar product**, and it has the same value in all inertial frames.

$$-\bar{a}^0 \bar{b}^0 + \bar{a}^1 b^1 + \bar{a}^2 \bar{b}^2 + \bar{a}^3 \bar{b}^3 = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

Just as the ordinary dot product is **invariant** under rotation, this combination is ***invariant under Lorentz transformation.***

We can introduce a **covariant** vector a_{μ}, which differs from the **contravariant** a^{μ} only in the sign of the zeroth component.

$$a_{\mu} = (a_0, a_2, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3)$$

Upper indices designate contravariant vectors and lower indices are for covariant vectors. Raising or lowering the temporal index costs a minus sign ($a_0 = -a^0$), but raising or lower spatial indices does not change anything ($a_1 = a^1, a_2 = a^2, a_3 = a^3$).

The scalar products can be written with the summation symbol:

$$\sum_{\mu=0}^{3} a_{\mu} b^{\mu}$$

Or more compactly:

$$a_{\mu} b^{\mu}$$

Summation is implied when the Greek index is repeated in a product-once as a covariant index and once as contravariant. This is called **Einstein summation convention**.

$$a_{\mu} b^{\mu} = a^{\mu} b_{\mu} = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3$$

(ii) The invariant interval

Suppose event A occurs at $(x_A^0, x_A^1, x_A^2, x_A^3)$ and event B occurs at $(x_B^0, x_B^1, x_B^2, x_B^3)$. The difference,

$$\Delta x^\mu = x_A^\mu - x_B^\mu$$

is the **displacement 4-vector**. The scalar product of Δx^μ with itself is a quantity of special importance; we call it the **interval** between two events:

$$I = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2 = -c^2t^2 + d^2$$

Where t is the difference between the two events and d is their spatial separation. When you transform to a moving system, the time between A and B is altered ($\bar{t} \neq t$) and so is the spatial separation ($\bar{d} \neq d$) but the interval I remains the same.

$$\bar{d} = \gamma(d - vt)$$

$$\bar{t} = \gamma\left(t - \frac{v}{c^2}d\right)$$

$$\bar{I} = -c^2\bar{t}^2 + \bar{d}^2 = -c^2\gamma^2\left[t^2 + \frac{v^2}{c^4}d^2 - 2t\frac{v}{c^2}d\right] + \gamma^2[d^2 + v^2t^2 - 2dvt]$$

$$\begin{aligned}\bar{I} &= -c^2\gamma^2t^2 - \frac{\gamma^2v^2}{c^2}d^2 + 2tv\gamma^2d + \gamma^2d^2 + \gamma^2v^2t^2 - 2dvt\gamma^2 \\ &= \gamma^2(v^2 - c^2)t^2 - \gamma^2\left(\frac{v^2}{c^2} - 1\right)d^2 \\ &= \frac{c^2}{(c^2 - v^2)}(v^2 - c^2)t^2 - \frac{c^2}{(c^2 - v^2)}\,(v^2 - c^2)\frac{d^2}{c^2}\end{aligned}$$

$$\bar{I} = -c^2t^2 + d^2 = I$$

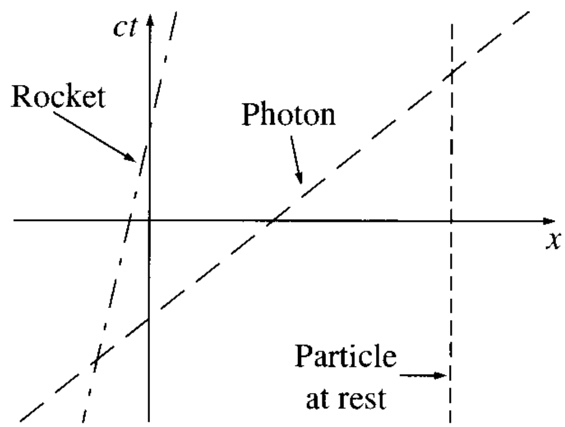
Depending on the two events in question, the interval can be positive, negative or zero:

1. If $I < 0$ we call the interval **timelike**, for this is the sign we get when the two events occur at the same place ($d = 0$) and are separated only temporally.
2. If $I > 0$ we call the interval **spacelike**, for this is the sign we get when the two events occur at the same time ($t = 0$) and are separated only spatially.
3. If $I = 0$ we call the interval **lightlike**, for this is the relation that holds when the two events are connected by a signal travelling at the speed of light.

If the interval between the events is **timelike**, there exists an inertial system (accessible by Lorentz transformation) where they occur at the same point. On the other hand, if the interval is **spacelike**, there exists a system in which the two events occur at the same time.

(iii) Space-time diagrams

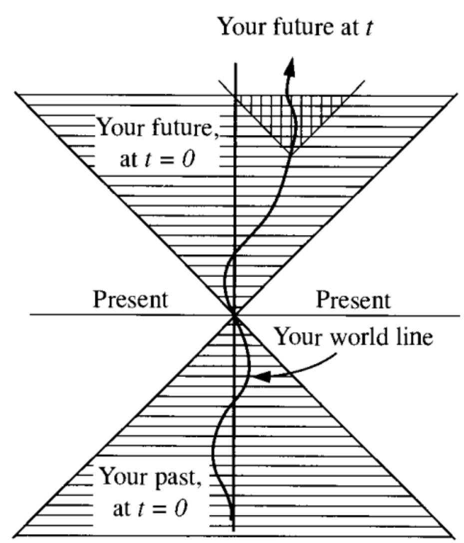
To represent the motion of a particle, a space-time graph is drawn. In relativity, time is drawn along the vertical axis and position is drawn along the horizontal line. A vertical line represents a particle at rest. A line with slope of 45° represent a particle moving with the speed of light and particle with intermediate speed is represented with the slope $\frac{c}{v} = \frac{1}{\beta}$. We call such plots **Minkowski diagrams**. The trajectory of a particle in Minkowski diagram is called a **world line**. Since no particle can move faster than the speed of light, so world line cannot have slope less than 1.



Our motion is restricted to the wedge-shaped region bounded by the two 45° degree lines. This is the locus of all points accessible to us. At any moment the forward edge is the **"future"** and the backward edge is the **"past"**.

In this drawing y and z-axes are ignored, if y-axis is included coming out of the page then it will represent a cone and if we include z-axis as well then it will become a hypercone.

The slope of the line connecting two events on the space-time diagram tells us whether the invariant interval between them is **timelike** (slope greater than 1) , **spacelike** (slope less than 1) or **lightlike** (slope=1)



If the invariant interval between two events is timelike, their ordering is absolute;

if the interval is spacelike, their ordering depends on the inertial system from which they are observed. Physics is based on the notion of **causality**, if it were possible to reverse the order of two events, then we could never say "A caused B" since a rival observer would retort that B preceded A.

Conclusion: ***The invariant interval between causally related events is always timelike, and their temporal ordering is the same for all inertial observers.***

12.2 Relativistic Mechanics

As one progresses along his world line, his watch runs slow, while the clock on the wall ticks off an interval dt, his watch only advances $d\tau$:

$$d\tau = \sqrt{1 - u^2/c^2}\, dt$$

Where u will be used as the velocity of the person (or particle) and v will be used for the relative velocity of two inertial systems. The time $\boldsymbol{\tau}$ associated with the moving objects is called **proper time**.

Proper time is invariant whereas **"ordinary"** time **t** depends on the particular reference frame.

If you are in a plane which is moving with velocity $\frac{4}{5}c$, then velocity means with respect to the ground, so

$$u = \frac{dl}{dt}$$

Where dl and dt are to be measured by the ground observer. However, the velocity measured by the observer on the plane is:

$$\eta = \frac{dl}{d\tau}$$

This hybrid quantity-distance measured on the ground but time measured in the plane frame is called **proper velocity** where we will call u the **ordinary velocity**.

The two velocities are related by the equation:

$$\eta = \frac{dl}{d\tau} = \frac{dl}{\sqrt{1 - u^2/c^2}\, dt} = \frac{u}{\sqrt{1 - u^2/c^2}}$$

Proper velocity has an enormous advantage over **ordinary velocity** when transforming from one inertial system to another because τ is invariant in any inertial system. In fact η is the spatial part of a 4-vector:

$$\eta^\mu = \frac{dx^\mu}{d\tau}$$

Whose zeroth component is:

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1 - u^2/c^2}}$$

When we go from system S to system $\bar{S}$, moving at speed v along the common $x\ \bar{x}$ axis,

$$\bar{\eta}^0 = \gamma(\eta^0 - \beta\eta^1)$$

$$\bar{\eta}^1 = \gamma(\eta^1 - \beta\eta^0)$$

$$\bar{\eta}^2 = \eta^2$$

$$\bar{\eta}^3 = \eta^3$$

Or more generally,

$$\bar{\eta}^\mu = \Lambda^\mu_\nu\, \eta^\nu$$

Where η^ν is called the **proper velocity 4-vector** , or simply the **4-velocity.**

By contrast, the transformation rule for ordinary velocities is extremely cumbersome

$$\bar{u}_x = \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{(1 - vu_x/c^2)}$$

$$\bar{u}_y = \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)}$$

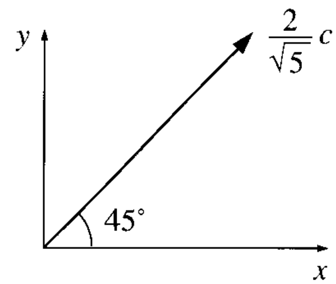
$$\bar{u}_z = \frac{d\bar{x}}{d\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)}$$

For **ordinary velocity** we need to transform both the numerator dl and the denominator dt. Whereas for **proper velocity** we only need to transform the numerator dl.

Example 6:

A car is travelling along the 45° line in S as shown in the figure below, at ordinary speed $(2/\sqrt{5})c$.

(a) Find the components u_x and u_y of the ordinary velocity.
(b) Find the components η_x and η_y of the proper velocity.
(c) Find the zeroth component of the 4-velocity, η^0



System $\bar{S}$ is moving in the x-direction with (ordinary) speed $(\sqrt{2/5})c$ relative to S. By using the appropriate transformation laws,

(d) Find the (ordinary) velocity components $\bar{u}_x$ and $\bar{u}_y$ in $\bar{S}$.
(e) Find the proper velocity components $\bar{\eta}_x$ and $\bar{\eta}_y$ in $\bar{S}$.
(f) As a consistency check, verify that:

$$\bar{\eta} = \frac{\bar{u}}{\sqrt{1 - \bar{u}^2/c^2}}$$

Solution:

a) $$u_x = u_y = u\cos 45° = \frac{1}{\sqrt{2}}\frac{2}{\sqrt{5}}c = \sqrt{\frac{2}{5}}\ c$$

b)

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{4}{5}}} = \sqrt{5}$$

$$\eta^\mu = \frac{dx^\mu}{d\tau} \rightarrow \eta^x = \frac{dx}{d\tau} = \frac{dx}{\sqrt{1-\frac{u^2}{c^2}}\,dt} = \frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{(\sqrt{2/5})\ c}{\sqrt{1-\frac{4c^2}{5}*\frac{1}{c^2}}} = \sqrt{2}c = \eta^y$$

c)

$$\eta^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-u^2/c^2}} = \frac{c}{\sqrt{1-\frac{4}{5}}} = \sqrt{5}\ c$$

d)

$$\bar{u}_x = \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{\left(1-\frac{vu_x}{c^2}\right)} = \frac{\sqrt{\frac{2}{5}}c - \sqrt{\frac{2}{5}}\ c}{\left(1-\frac{2}{5}\right)} = 0$$

$$\bar{u}_y = \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1-vu_x/c^2)} = \sqrt{1-u^2/c^2}\,\frac{\sqrt{\frac{2}{5}}c}{\left(1-\frac{2}{5}\right)} = \sqrt{1-\frac{2}{5}}\frac{\sqrt{\frac{2}{5}}c}{\left(1-\frac{2}{5}\right)} = \sqrt{\frac{2}{3}}c$$

e)

$$\bar{\eta}^x = \gamma(\eta^x - \beta\eta^0) = \sqrt{1-\frac{2}{5}}\left(\sqrt{2} - \frac{v}{c}\frac{c}{\sqrt{1-u^2/c^2}}\right) = \sqrt{\frac{3}{5}}\left(\sqrt{2}c - \sqrt{\frac{2}{5}}\ c * \frac{1}{\sqrt{1-\frac{4}{5}}}\right) = 0 = \eta^x$$

$$\bar{\eta}^y = \eta^y = \sqrt{2}\ c$$

f)

$$\bar{\eta} = \frac{1}{\sqrt{1-\bar{u}^2/c^2}}\bar{u} = \frac{1}{\sqrt{1-\frac{2}{3}}}\bar{u} = \sqrt{3}\ \bar{u}$$

$$\bar{\eta}^x = \sqrt{3}\ \bar{u}^x = 0$$

$$\bar{\eta}^y = \sqrt{3}\,\bar{u}^y = \sqrt{3} * \sqrt{\frac{2}{3}}c = \sqrt{2}c$$

Which are same values as we found in part (e)

12.2.2 Relativistic Energy and Momentum

Momentum is mass times velocity but what velocity we should use, ordinary velocity or proper velocity. The answer is we need to use proper velocity because law of conservation of momentum would be inconsistent with the principle of relativity if we use ordinary velocity. Hence

$$\vec{p} = m\vec{\eta} = \frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}$$

This is the **relativistic momentum** and this is spatial part of a 4-vector.

$$p^\mu = m\eta^\mu$$

And the temporal component is:

$$p^0 = m\eta^0 = \frac{mc}{\sqrt{1 - u^2/c^2}}$$

Einstein called:

$$m_{rel} = \frac{m}{\sqrt{1 - u^2/c^2}}$$

The **relativistic mass** (so that $p^0 = m_{rel}c$ and $\vec{p} = m_{rel}\vec{u}$ and m itself is called the rest mass) But now we just use energy terminology instead:

$$E = p^0 c = \frac{mc^2}{\sqrt{1 - u^2/c^2}}$$

$$p^0 = E/c$$

Notice that relativistic energy is non-zero even if the particle is not moving, it is called **rest energy**.

$$E_{rest} = mc^2$$

And the remainder which is attributed to motion, is called **kinetic energy**.

$$E_{kin} = E - mc^2 = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right)$$

In the non-relativistic regime ($u \ll c$), the square root can be expanded in powers of u^2/c^2 giving:

$$E_{kin} = mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} + \frac{3}{8}\frac{u^4}{c^4} + \cdots - 1\right) = \frac{1}{2}mu^2 + \frac{3}{8}\frac{mu^4}{c^2} + \cdots$$

The leading term reproduces the classical formula.

In every closed system, the total relativistic energy and momentum are conserved.

(Rest mass is not conserved because of its conversion to energy)

We have to understand the distinction between the **invariant** quantity (same value in all inertial frames) and **conserved** quantity (same value before and after a process). For example, mass is invariant but not conserved, energy is conserved but not invariant, velocity is neither conserved nor invariant.

The scalar product of p^μ with itself gives:

$$p^\mu p_\mu = -(p^0)^2 + (\vec{p}.\vec{p}) = -m^2c^2$$

In terms of the relativistic energy:

$$E^2 - p^2c^2 = m^2c^4$$

This result is useful because we can calculate E if we know p and vice versa.

Example 7:

Consider that in an inertial frame S, a particle A (mass m_A , velocity $\vec{u}_A$) hits particle B (mass m_B , velocity $\vec{u}_B$) and they are moving in the opposite direction $\vec{u}_A = -\vec{u}_B$. Suppose it is a completely inelastic collision where the two particles stick after the collision and come to rest.

(a) Prove that using incorrect definition of momentum $\vec{p} = m\vec{u}$ but with correct Einstein velocity addition rule, momentum is conserved in S but not conserved in $\bar{S}$ which is another inertial from moving with velocity $\vec{v}$ with respect to S.
(b) Now use the correct definition of momentum $\vec{p} = m\vec{\eta}$ and see if the momentum is conserved in both S and $\bar{S}$. What must you assume about relativistic energy?

Solution:

(a)

$$m_A u_A + m_B u_B = m_C u_C + m_D u_D$$

According to Einstein's velocity addition rule:

$$u_i = \frac{\bar{u}_i + v}{1 + (\bar{u}_i v/c^2)}$$

$$m_A \frac{\bar{u}_A + v}{1 + (\bar{u}_A v/c^2)} + m_B \frac{\bar{u}_B + v}{1 + (\bar{u}_B v/c^2)} = m_C \frac{\bar{u}_C + v}{1 + (\bar{u}_C v/c^2)} + m_D \frac{\bar{u}_D + v}{1 + (\bar{u}_D v/c^2)}$$

Now suppose all the masses are equal and $u_A = -u_B = v$ and $u_C = u_D = 0$. This is a completely inelastic collision in S and momentum is conserved.

But according to Einstein's velocity addition rule:

$$\bar{u}_i = \frac{v - u_i}{\frac{u_i v}{c^2} - 1}$$

$\bar{u}_A = 0$ and $\bar{u}_B = -\frac{2v}{(1+v^2/c^2)}$, $\bar{u}_C = \bar{u}_D = -v$

So in frame $\bar{S}$, the momentum before and after the collision is:

$$m\bar{u}_A + m\bar{u}_B = m\bar{u}_C + m\bar{u}_D$$

$$0 - \frac{2v}{\left(1 + \frac{v^2}{c^2}\right)} = -2v$$

Which is not conserved.

(b) Now using the correct definition of relativisitic momentum and use proper velocity:

$$m_A \eta_A + m_B \eta_B = m_C \eta_C + m_D \eta_D$$

Where according to Lorentz transformation:

$$\eta_i = \gamma(\bar{\eta}_i + \beta \bar{\eta}_i^0)$$

Hence in $\bar{S}$ frame:

$$m_A \gamma(\bar{\eta}_A + \beta \bar{\eta}_A^0) + m_B \gamma(\bar{\eta}_B + \beta \bar{\eta}_B^0) = m_C \gamma(\bar{\eta}_C + \beta \bar{\eta}_C^0) + m_D \gamma(\bar{\eta}_D + \beta \bar{\eta}_D^0)$$

$$m_A \bar{\eta}_A + m_B \bar{\eta}_B + \beta(m_A \bar{\eta}_A^0 + m_B \bar{\eta}_B^0) = m_C \bar{\eta}_C + m_D \bar{\eta}_D + \beta(m_C \bar{\eta}_C^0 + m_D \bar{\eta}_D^0)$$

But $m_i \bar{\eta}_i^0 = E_i/c$ so if energy is conserved, then

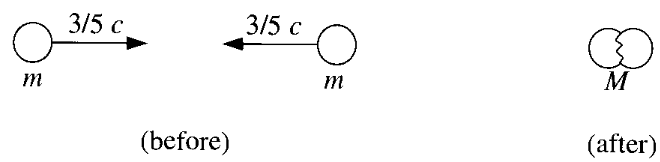
$$m_A \bar{\eta}_A + m_B \bar{\eta}_B = m_C \bar{\eta}_C + m_D \bar{\eta}_D$$

Momentum is conserved as well in $\bar{S}$.

12.2.3 Relativistic Kinematics

Example 8:

Two lumps of clay, each of (rest) mass m, collide head-on at $\frac{3}{5}c$ as depicted in the figure below. They stick together after the collision. What is the mass (M) of the composite lump?



Solution:

In this case conservation of momentum is simple, zero before and zero after. The energy of each lump prior to the collision is:

$$\frac{mc^2}{\sqrt{1-\left(\frac{3}{5}\right)^2}} = \frac{5}{4}mc^2$$

The energy of the composite lump after the collision is Mc^2 because it is at rest.

$$Mc^2 = \frac{5}{4}mc^2 + \frac{5}{4}mc^2 = \frac{5}{2}mc^2$$

$$M = \frac{5}{2}m$$

Note that mass after the collision is greater than before the collision, mass is not conserved because kinetic energy was converted into rest energy, hence the mass increased.

In classical mechanics we cannot think about massless particle with energy and momentum because they will be zero. But in relativity we can have a massless particle with energy and momentum if it is moving with a speed of light then both denominator and numerator of the momentum would be zero leaving it indeterminate.

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$

So a massless particle can have energy and momentum as long as it is moving with the speed of light. One example is photons that are massless and move with the speed of light but carry energy and momentum. Although relativity does not tell us why different photons have different energy when they have the same mass (zero) and same speed (c). This we know from quantum mechanics, where Plank found that energy of a photon is dependent on its frequency:

$E = h\nu$ where h is plank's constant. [A blue photon is more energetic than a red photon]

Example 09: Compton Scattering: A photon of energy E_o "bounces" off an electron, initially at rest. Find the energy E of the outgoing photon, as a function of the scattering angle θ.

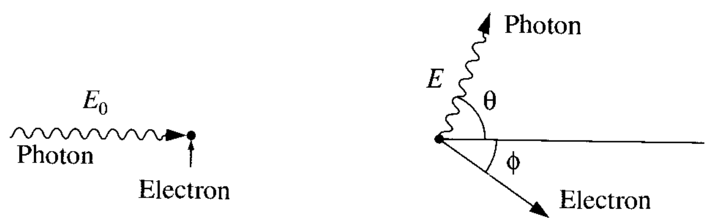
Solution:

Conservation of momentum in the vertical direction gives:

$$p_e \sin\phi = p_p \sin\theta$$

Since $p_p = \frac{E}{c}$

$$\sin\phi = \frac{E}{p_e c}\sin\theta$$



(before) (after)

Conservation of momentum in the horizontal direction gives:

$$\frac{E_o}{c} = p_p\cos\theta + p_e\cos\phi$$

$$= \frac{E}{c}\cos\theta + p_e\sqrt{1-\left(\frac{E}{p_e c}\sin\theta\right)^2}$$

$$p_e^2c^2 = (E_o - E\cos\theta)^2 + E^2\sin^2\theta = E_o^2 - 2E_oE\cos\theta + E^2$$

Finally, conservation of energy says that:

$$E_o + mc^2 = E + E_e = E + \sqrt{m^2c^4 + p_e^2c^2} = E + \sqrt{m^2c^4 + E_o^2 - 2E_oE\cos\theta + E^2}$$

$$E = \frac{1}{\frac{(1-\cos\theta)}{mc^2} + \left(\frac{1}{E_o}\right)}$$

If we express it interms of photon wavelength:

$$E = h\nu = \frac{hc}{\lambda}$$

$$\lambda = \lambda_o + \frac{h}{mc}(1-\cos\theta)$$

The quantity (h/mc) is called the **Compton wavelength** of the electron.

12.2.4 Relativistic Dynamics

Newton's second law is valid in the relativistic mechanics as long as the momentum is considered relativistic momentum,

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Example 10:

Motion under a constant force: A particle of mass m is subject to a constant force F. If it starts from rest at the origin, at time t=0, find its position (x) as a function of time.

Solution:

$$\frac{dp}{dt} = F \rightarrow p = Ft + constant$$

But since $p = o$ at $t = 0$, the constant must be zero, and hence:

$$p = \frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = Ft$$

Solving for u, we obtain:

$$u = \frac{(F/m)t}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}}$$

The numerator is the classical answer if $\left(\frac{F}{m}\right) t \ll c$. But the relativistic denominator ensures that u never exceeds ; in fact as $t \rightarrow \infty$, $u \rightarrow c$.

To complete the problem we must integrate again:

$$x(t) = \frac{F}{m}\int_0^t \frac{t'}{\sqrt{1 + \left(\frac{Ft'}{mc}\right)^2}} dt' = \frac{mc^2}{F}\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}\Bigg|_o^t = \frac{mc^2}{F}\left[\sqrt{1 + \left(\frac{Ft}{mc}\right)^2} - 1\right]$$

In place of a classical parabola, $x = (F/2m)t^2$,the graph is hyperbola; for this reason motion under a constant force is often called hyperbolic motion. It occurs for example, when a charged particle is placed in a uniform electric field.

Work done by the field is the line integral of the force:

$$W = \int \vec{F}.d\vec{l}$$

The work-energy theorem holds relativistically as well:

$$W = \int \frac{d\vec{p}}{dt}.d\vec{l} = \int \frac{d\vec{p}}{dt}.\frac{d\vec{l}}{dt}dt = \int \frac{d\vec{p}}{dt}.\vec{u}\,dt$$

$$\frac{d\vec{p}}{dt}.\vec{u} = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right) = \frac{dE}{dt}$$

$$W = \int \frac{dE}{dt}dt = E_{final} - E_{initial}$$

Since the rest energy is constant so $W = K_{final} - K_{initial}$

Newton's third law does not extend to relativistic domain. If the two objects are separated in space, the third law is incompatible with the relativity of simultaneity. For suppose the force of A onto object B at some instant t is $F(t)$, and the force of B on A at the same instant is $-F(t)$; then the third law applies in this frame. But a moving observer will report that these equal and opposite forces did not occur at the same time, so in his frame the third law is violated.

Since $\vec{F}$ is the derivative of momentum with respect to ordinary time, it shares the ugly behavior of "ordinary" velocity when you go from one inertial system to another: both the numerator and denominator must be transformed. Thus

$$\bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{dp_y/dt}{\gamma\left(1-\frac{\beta}{c}\frac{dx}{dt}\right)} = \frac{F_y}{\gamma\left(1-\frac{\beta}{c}u_x\right)}$$

Similarly for the z-component:

$$\bar{F}_z = \frac{F_z}{\gamma\left(1-\frac{\beta}{c}u_x\right)}$$

The x-component will be:

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma dp_x - \gamma\beta dp^0}{\gamma dt - \frac{\gamma\beta}{c}dx} = \frac{\frac{dp_x}{dt} - \beta\frac{dp^0}{dt}}{1-\frac{\beta}{c}\frac{dx}{dt}} = \frac{F_x - \frac{\beta}{c}\left(\frac{dE}{dt}\right)}{1-\frac{\beta}{c}u_x}$$

Where

$$\frac{dE}{dt} = \frac{d\vec{p}}{dt}.\vec{u} = \vec{F}.\vec{u}$$

$$\bar{F}_x = \frac{F_x - \frac{\beta}{c}(\vec{F}.\vec{u})}{1 - \frac{\beta}{c} u_x}$$

If the particle is (instantaneously) at rest in S, so that $\vec{u} = 0$, then

$$\bar{F}_\perp = \frac{1}{\gamma} F_\perp$$
$$\bar{F}_\parallel = F_\parallel$$

The component of $\vec{F}$ parallel to the motion of $\bar{S}$ is unchanged, whereas the components perpendicular are divided by γ.

We could avoid this ugly transformation by introducing a "proper" force, analogous to proper velocity, which would be the derivative of momentum with respect to *proper time*.

$$K^\mu \equiv \frac{dp^\mu}{d\tau}$$

This is called the **Minkowski force**; it is plainly a 4-vector, since p^μ is a 4-vector and proper time is invariant. The spatial component of K^μ are related to the "ordinary" force by:

$$\vec{K} = \left(\frac{dt}{d\tau}\right)\frac{d\vec{p}}{dt} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\vec{F}$$

While the zeroth component,

$$K^0 \equiv \frac{dp^0}{d\tau} = \frac{1}{c}\frac{dE}{d\tau} \quad \text{where} \quad p^0 = E/c$$

It is the (proper) rate at which the energy of the particle increases or the (proper) power delivered to the particle.

Relativistic dynamics can be formulated in terms of the **ordinary force** or in terms of the **Minkowski force**.

Example 11:

Hidden Momentum: As a model for a magnetic dipole **m**, consider a rectangular loop of wire carrying a steady current. Picture the current as a stream of noninteracting positive charges that move freely within the wire. When a uniform electric field $\vec{E}$ is applied as shown in the figure below, the charges accelerate in the left segment and decelerate in the right one. Find the total momentum of all the charges in the loop.

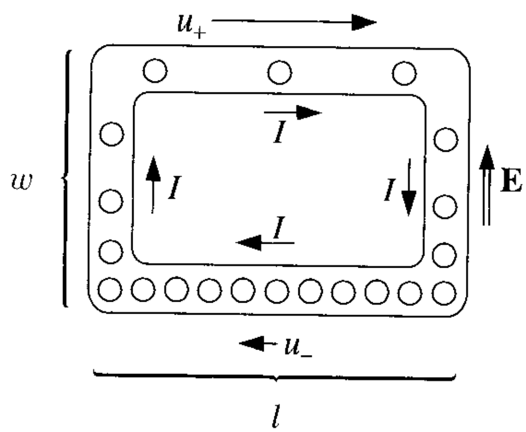
Solution:

Momenta of the left and right segments cancel, so we need only consider the top and the bottom segments. Suppose there are N+ charges in the top segment going at speed u+ to the right and N- charges in the lower segment going at (slower) speed u- to the left.

The current is the same in all segments ($I = \lambda u$) else charge would be piling up somewhere.

$$I = \frac{QN_+}{l}u_+ = \frac{QN_-}{l}u_-$$

$$N_\pm u_\pm = \frac{Il}{Q}$$



Where Q is the charge of each particle and l is the length of the rectangle. Classically total momentum is:

$$p_{classical} = MN_+u_+ - MN_-u_- = M\frac{Il}{Q} - M\frac{Il}{Q}$$
$$= 0$$

As one would expect because the loop as a whole is not moving. But relativisitically:

$$\vec{p} = \gamma M\vec{u}$$

And we get:

$$p = \gamma_+ MN_+u_+ - \gamma_- MN_-u_- = \frac{MIl}{Q}(\gamma_+ - \gamma_-)$$

Which is not zero, because the particle in the upper segment are moving faster. In fact the gain in energy (γMc^2) as a particle goes up the left segment, is equal to the work done by the electric force, QEw where w is the height of the loop, so

$$\gamma_+ - \gamma_- = \frac{QEw}{Mc^2}$$

$$p = \frac{MIl}{Q}(\gamma_+ - \gamma_-) = \frac{MIl}{Q}\frac{QEw}{Mc^2} = \frac{IlEw}{c^2}$$

Where Ilw is the dipole moment of the loop, as vectors $\vec{m}$ points into the page and $\vec{p}$ is to the right, so

$$\vec{p} = \frac{1}{c^2}(\vec{m} \times \vec{E})$$

Thus a magnetic dipole in an electric field carries linear momentum, *even though it is not moving!* This so called **hidden momentum** is strictly relativistic, and purely mechanical; it precisely cancels the electromagnetic momentum stored in the fields.

12.3 Relativistic Electrodynamics

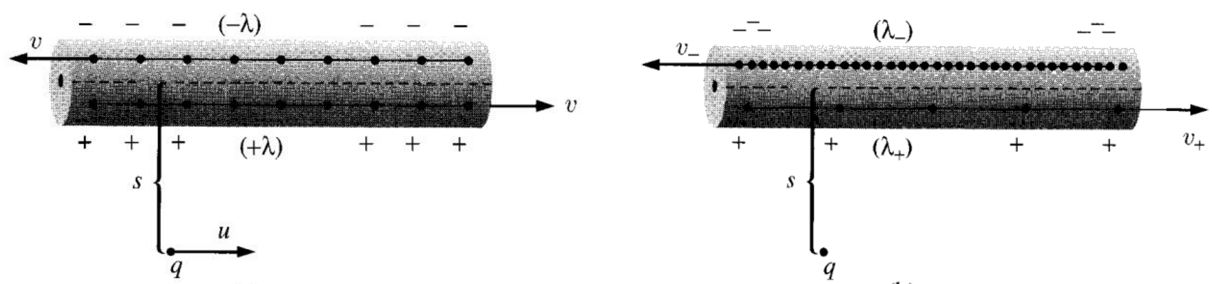
12.3.1 Magnetism as a Relativistic Phenomenon

Unlike Newtonian mechanics, classical electrodynamics is already consistent with special relativity. Maxwell's equations and the Lorentz force law can be applied legitimately in an inertial system. Of course, what one observer interprets as an electrical process another may regards as magnetic, but the actual particle motions they predict will be identical.

So in a way, if we used electrostatics and relativity, we can calculate the magnetic force between a current-carrying wire and a moving charge without even invoking the laws of magnetism.

Suppose there is a string of positive charges moving along to the right at speed v and lets assume that charges are close enough that they form a continuous line charge λ. Superimposed on this positive string is a negative one $-\lambda$ proceeding to the left at the same speed v. So the net current is to the right:

$$I = 2\lambda v$$



(a) (b)

And distance s away from the line of charges there lies a charge q travelling to the right at speed $u < v$.

Since the two line charges cancel each other so there is no electrical force on charge q in this system S.

Lets assume there is another fram $\bar{S}$, which moves to the right with speed u. In this reference frame q is at rest. By the Eisntein's velocity addition rule, the velocities of the positive and negative lines are now:

$$v_+ = \frac{v-u}{1-vu/c^2} \quad \text{and} \quad v_- = \frac{v+u}{1+vu/c^2}$$

Because v_- is greater than v_+, the Lorentz contraction of the spacing between the negative charges will be more severe than between the positive charges, in this frame, therefore, the wire carries a net negative charge, in fact:

$$\lambda_+ = \gamma_+\lambda_o \quad \text{and} \quad \lambda_- = -\gamma_-\lambda_o$$

Where

$$\gamma_+ = \frac{1}{\sqrt{1-v_+^2/c^2}} \quad \text{and} \quad \gamma_- = \frac{1}{\sqrt{1-v_-^2/c^2}}$$

And λ_o is the charge density of the positive line in its own rest system and this is not the same as λ, of course – in S they are already moving at speed v, so

$$\lambda = \gamma\lambda_o$$

Where

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\gamma_+ = \frac{1}{\sqrt{1-v_+^2/c^2}} = \frac{1}{\sqrt{1-\left(\frac{v-u}{1-\frac{vu}{c^2}}\right)^2 \frac{1}{c^2}}} = \frac{1}{\sqrt{1-c^2\left(\frac{v-u}{c^2-vu}\right)^2}}$$

$$\gamma_+ = \frac{c^2-vu}{\sqrt{(c^2-vu)^2-c^2(v-u)^2}} = \frac{c^2-vu}{\sqrt{(c^2-v^2)(c^2-u^2)}} = \gamma\frac{1-uv/c^2}{\sqrt{1-u^2/c^2}}$$

Similarly

$$\gamma_- = \gamma\frac{1+uv/c^2}{\sqrt{1-u^2/c^2}}$$

Evidently, then, the net line charge in $\bar{S}$ is:

$$\lambda_{total} = \lambda_+ + \lambda_- = \lambda_o(\gamma_+ - \gamma_-) = \frac{\lambda}{\gamma}\left(\gamma\frac{1-uv/c^2}{\sqrt{1-u^2/c^2}} - \gamma\frac{1+uv/c^2}{\sqrt{1-u^2/c^2}}\right) = -\frac{2\lambda uv}{c^2\sqrt{1-u^2/c^2}}$$

Conclusion: ***As a result of unequal Lorentz contraction of the positive and negative lines, a current-carrying wire that is electrically neutral in one inertial system will be charged in another.***

Now a line charge λ_{total} sets up an electric field:

$$E = \frac{\lambda_{tot}}{2\pi\epsilon_o s}$$

So, there is an electric force on q in $\bar{S}$, which is:

$$\bar{F} = qE = -\frac{\lambda v}{\pi\epsilon_o c^2 s}\frac{qu}{\sqrt{1-u^2/c^2}}$$

But is there is a force on q in $\bar{S}$ then there must be a force in S as well, in fact we can calculate using the transformation rules for the forces. Since q is at rest in $\bar{S}$ and $\bar{F}$ is perpendicular to u, the force in S is given by:

$$\bar{F}_\perp = \frac{1}{\gamma}F_\perp$$

$$F = \sqrt{1-u^2/c^2}\,\bar{F} = \sqrt{1-u^2/c^2} * \left[-\frac{\lambda v}{\pi\epsilon_o c^2 s}\frac{qu}{\sqrt{1-\frac{u^2}{c^2}}}\right] = -\frac{\lambda v}{\pi\epsilon_o c^2}\frac{qu}{s}$$

Using $c^2 = (\epsilon_o\mu_o)^{-1}$ and $\lambda v = I$

$$F = -qu\left(\frac{\mu_o I}{2\pi s}\right)$$

The term in parenthesis is the magnetic field produced by a long straight wire and the force on charge q moving with speed u is the Lorentz force in S.

The charge is attracted towards the wire by a force that is purely electrical in $\bar{S}$ (where the wire is charged and q is at rest) but distinctly nonelectrical (in fact magnetic) in S (where the wire is neutral and charge is moving with speed u).

12.3.2 How the Field Transform