1 Probability Distributions

We have three distributions, one for classical particles (Maxwell-Boltzmann) and two for undistinguishable particles (Fermi-Dirac for fermions and Bose-Einstein for bosons). Their equations are the following:

$$P(E) = \frac{1}{e^{\beta(E-E_f)}+1} \qquad \text{Fermi–Dirac Dist.} \tag{1}$$

$$P(E) = \frac{1}{e^{\beta(E-\mu)}-1} \qquad \text{Bose–Einstein Dist.} \tag{2}$$

$$P(E) = \frac{1}{e^{\beta(E-\mu)}} \qquad \text{Maxwell–Boltzmann Dist.} \tag{3}$$

$$\text{Where } \beta = \frac{1}{k_B T} \tag{4}$$

Now I will plot these distributions for different temperatures:

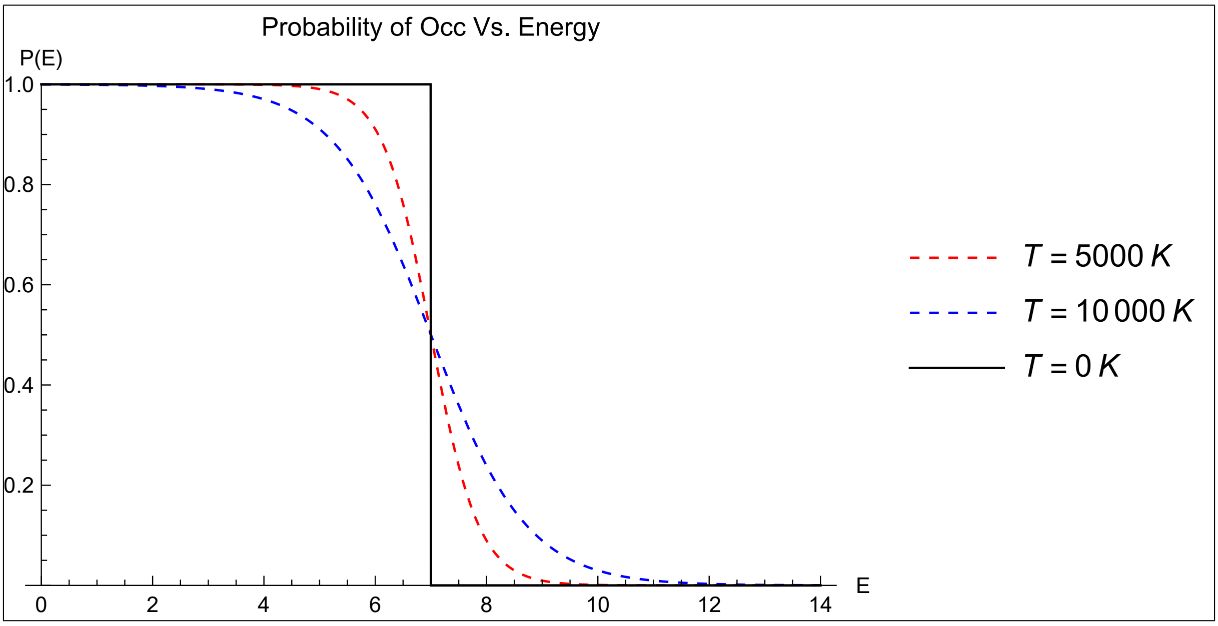

Figure 1: Fermi–Dirac Dist. when $E_f = 7eV$. For fermions

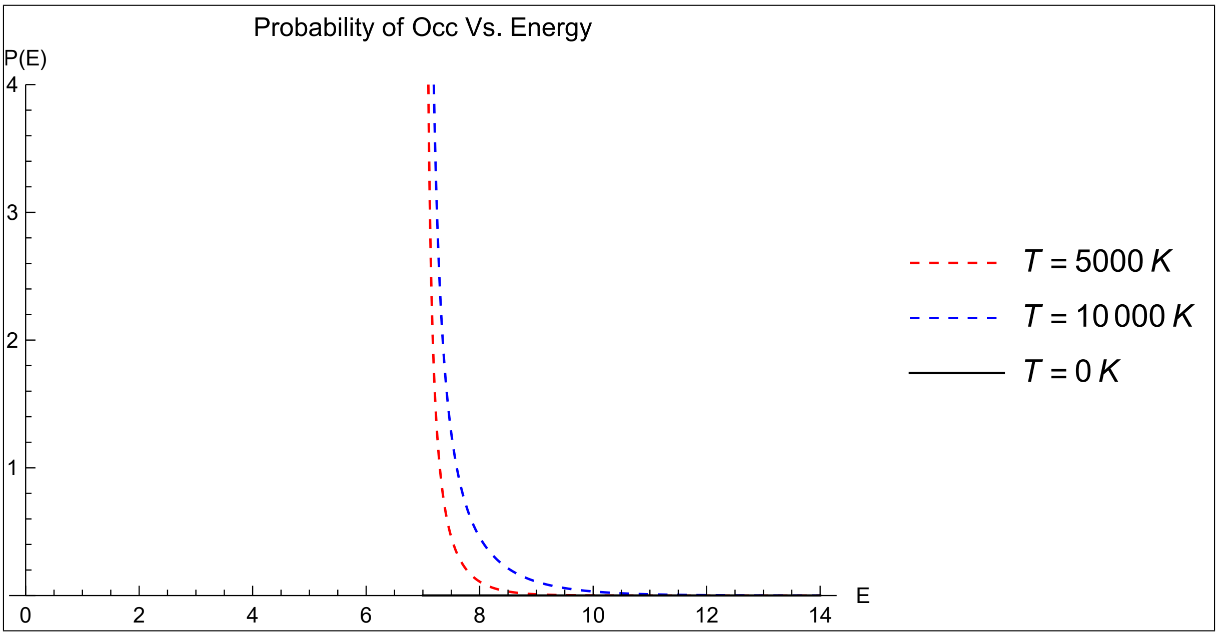

Figure 2: Bose–Einstein Dist. when $\mu = 7eV$. For bosons

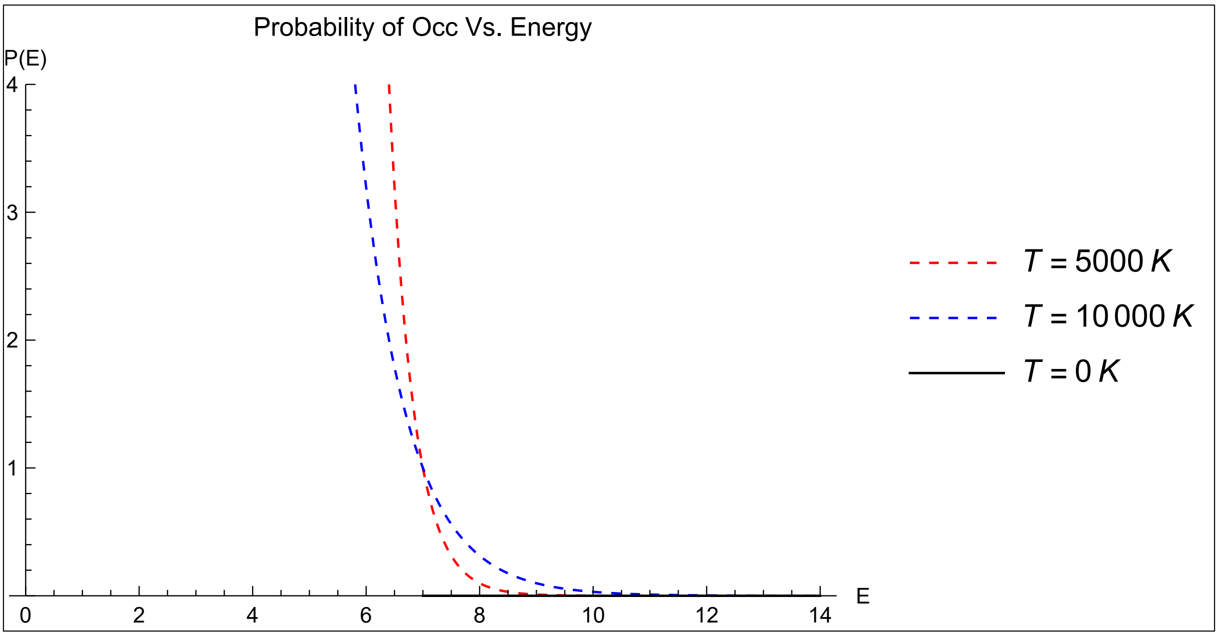

Figure 3: Maxwell–Boltzmann Dist. when $\mu = 7eV$. For bosons

2 Remarks

Fermions are subject to Fermi–Dirac statistics. We can observe that no state will be occupied by any more than 1 fermion. This is because fermions obey Pauli's Exclusion Principle which states that no two fermions can occupy the same quantum state. However, since bosons do not obey Pauli's Exclusion Principle, they can occupy the same state as we can see in the above plot and they are subject to Bose–Einstein statistics.

Moreover, when fermions are at temperatures higher than $0K$, some fermions will leak and have some probability to occupy states after the Fermi level. However, at the Fermi level, fermions will always have a probability of $\frac{1}{2}$ regardless of temperature.

3 Examples

Ex1: Consider a metal with one conduction electron per atom, for example, Gold with a density of $19.3g/cm^3$. Calculate k_F, E_F, T_F, λ_F Fermi wavenumber, energy, temperature, and Wavelength, respectively, along with the average spacing r_S. Then compare at room temperature $T = 300K$ the ratios λ_F/r_S and T/T_F and explain their physical meaning.

First we need to find the electronic density n, from it we can extract k_F and r_S directly

$$n = \frac{19.3\cancel{g}}{cm^3} \times \frac{\cancel{mol}}{196.96657\cancel{g}} \times (\#\text{ of Cond. Ele.} = 1) \times N_A = 5.9 \times 10^{22} electron/cm^3 = 5.9 \times 10^{28} electron/m^3$$

$$k_F = (3\pi^2 n)^{1/3} = (3\pi^2 \times 5.9 \times 10^{28})^{1/3} = 1.2 \times 10^{10} m^{-1}$$

$$r_S = \left(\frac{3}{4\pi n}\right)^{1/3} = \left(\frac{3}{4\pi 5.9 \times 10^{28}}\right)^{1/3} = 1.59 \times 10^{-10} m = 1.59\text{Å}$$

From k_F we can directly get E_F :

$$E_F = \frac{\hbar^2 k_F^2}{2m_e} = 8.8 \times 10^{-19} J \approx 5.5eV$$

$$\lambda_F = \frac{2\pi}{k_F} = 5.24 \times 10^{-10} m = 5.24\text{Å}$$

$$T_F = \frac{E_F}{k_B} = 6.38 \times 10^4 K$$

Now let's calculate the ratios λ_F/r_S and T/T_F:

$$\lambda_F/r_S = 3.3 \qquad T/T_F = 4.7 \times 10^{-3}$$

Since $\lambda_F/r_S > 1$, we have to treat the electrons as indistinguishable particles because their wavefunctions will overlap. k_BT is the range of variation of the Fermi-Dirac Distribution. The ratio T/T_F is the ratio of thermal energy to fermi energy k_BT/E_F which is very low

Ex2: Redo Ex.1 but for a metal with two conduction electrons, e.g., Iron with a density of $7.86g/cm^3$:

$$n = \frac{7.86\cancel{g}}{100^{-3}m^3} \times \frac{\cancel{mol}}{55.845\cancel{g}} \times (\#\text{ of Cond. Ele.} = 2) \times N_A = 1.7 \times 10^{29} electron/m^3$$

$$k_F = (3\pi^2 n)^{1/3} = (3\pi^2 \times 1.7 \times 10^{29})^{1/3} = 1.7 \times 10^{10} m^{-1}$$

$$r_S = \left(\frac{3}{4\pi n}\right)^{1/3} = \left(\frac{3}{4\pi 1.7 \times 10^{29}}\right)^{1/3} = 1.12 \times 10^{-10} m = 1.12\text{Å}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m_e} = 1.78 \times 10^{-18} J \approx 11.2eV$$

$$\lambda_F = \frac{2\pi}{k_F} = 3.67 \times 10^{-10} m = 3.37\text{Å}$$

$$T_F = \frac{E_F}{k_B} = 13.0 \times 10^4 K$$

$$\lambda_F/r_S = 3.00 \qquad T/T_F = 2.3 \times 10^{-3}$$