Solution Major Exam I

Phys. 310

Q1.

(a) $$\Psi(x,0) = \frac{A}{\sqrt{12}}\phi_1(x) + \frac{1}{\sqrt{6}}\phi_2(x) + \frac{1}{\sqrt{3}}\phi_3(x) + \frac{1}{2}\phi_4(x)$$

$$\int_{-\infty}^{+\infty}\Psi^*(x,0)\,\Psi(x,0)\,dx = \frac{A^2}{12}\int\phi_1^*\phi_1\,dx + \frac{1}{6}\int\phi_2^*\phi_2\,dx + \frac{1}{3}\int\phi_3^*\phi_3\,dx + \frac{1}{4}\int\phi_4^*\phi_4\,dx$$

$$1 = \frac{A^2}{12} + \frac{1}{6} + \frac{1}{3} + \frac{1}{4} \qquad \text{Since } \int\phi_m^*\phi_n\,dx = \delta_{m,n}$$

$$\Rightarrow \frac{A^2}{12} = \frac{1}{4} \Rightarrow A^2 = 3 \Rightarrow A = \sqrt{3}$$

Thus $$\Psi(x,0) = \frac{1}{2}\phi_1(x) + \frac{1}{\sqrt{6}}\phi_2(x) + \frac{1}{\sqrt{3}}\phi_3(x) + \frac{1}{2}\phi_4(x)$$

(b) $$\Psi(x,t) = \frac{1}{2}e^{-iE_1t/\hbar}\phi_1(x) + \frac{1}{\sqrt{6}}e^{-iE_2t/\hbar}\phi_2(x) + \frac{1}{\sqrt{3}}e^{-iE_3t/\hbar}\phi_3(x) + \frac{1}{2}e^{-iE_4t/\hbar}\phi_4(x)$$

$$\Psi(x,t) = e^{-i\omega t/2}\left[e^{-i\omega t}\phi_1(x)\frac{1}{2} + \frac{1}{\sqrt{6}}e^{-2i\omega t}\phi_2(x) + \frac{1}{\sqrt{3}}e^{-i3\omega t}\phi_3(x) + \frac{1}{2}e^{-i4\omega t}\phi_4(x)\right]$$

$$\Psi(x,t) = e^{-i\frac{3}{2}\omega t}\left[\frac{1}{2}\phi_1(x) + \frac{1}{\sqrt{6}}e^{-i\omega t}\phi_2(x) + \frac{1}{\sqrt{3}}e^{-i2\omega t}\phi_3(x) + \frac{1}{2}e^{-i3\omega t}\phi_4(x)\right]$$

(c) $$E = \int\Psi^*(x,t)\,H\,\Psi(x,t)\,dx = \int\Psi^*(x,0)\,H\,\Psi(x,0)\,dx$$

$$= \int\left(\frac{1}{2}\phi_1 + \frac{1}{\sqrt{6}}\phi_2 + \frac{1}{\sqrt{3}}\phi_3 + \frac{1}{2}\phi_4\right)\left(\frac{E_1}{2}\phi_1 + \frac{E_2}{\sqrt{6}}\phi_2 + \frac{E_3}{\sqrt{3}}\phi_3 + \frac{E_4}{2}\phi_4\right)dx$$

$$= \frac{E_1}{4} + \frac{E_2}{6} + \frac{E_3}{3} + \frac{E_4}{4} = \frac{1}{24}\left(6E_1 + 4E_2 + 8E_3 + 6E_4\right)$$

$$E = \frac{\hbar\omega}{24}\left[6\left(\frac{3}{2}\right) + 4\left(\frac{5}{2}\right) + 8\left(\frac{7}{2}\right) + 6\left(\frac{9}{2}\right)\right] = \frac{37}{12}\hbar\omega$$

Q2.

(a) $$E = \int\Psi^*(x)\,H\,\Psi(x)\,dx = \int_{-\infty}^{+\infty}\Psi^*(x)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\Psi(x)\,dx$$

$$E = -\frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\Psi^*(x)\frac{d}{dx}\left(\frac{d\Psi}{dx}\right)dx + \int\Psi^*(x)\,V(x)\,\Psi(x)\,dx$$

$$E = -\frac{\hbar^2}{2m}\left\{\Psi^*(x)\frac{d\Psi}{dx}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}\left|\frac{d\Psi}{dx}\right|^2dx\right\} + \int\Psi^*(x)\,V(x)\,\Psi(x)\,dx$$

$$E = \frac{\hbar^2}{2m}\int_{-\infty}^{+\infty}\left|\frac{d\Psi}{dx}\right|^2dx + \int\Psi^*(x)\,V(x)\,\Psi(x)\,dx \geqslant V_{min}\int\Psi^*\Psi\,dx = V_{min}$$

notice that the first quantity is >0.

(b)

$$[x,p]F(x) = \left(x\frac{\hbar}{i}\frac{d}{dx} - \frac{\hbar}{i}\frac{d}{dx}x\right)F(x) \quad ; \forall F(x)$$

$$[x,p]F(x) = x\frac{\hbar}{i}\frac{dF}{dx} - \frac{\hbar}{i}F(x) - \frac{\hbar}{i}x\frac{dF}{dx} = -\frac{\hbar}{i}F(x) = i\hbar F(x)$$

$$\Rightarrow [x,p] = i\hbar \quad ; \quad p = \frac{\hbar}{i}\frac{d}{dx} \Rightarrow p^2 = -\hbar^2\frac{d^2}{dx^2}$$

$$[x, \frac{d^2}{dx^2}] = -\frac{1}{\hbar^2}[x,p^2] = -\frac{1}{\hbar^2}\{[x,p]p + p[x,p]\}$$

$$[x, \frac{d^2}{dx^2}] = -\frac{1}{\hbar}\{i\hbar p + i\hbar p\} = -2ip/\hbar$$

$$[x^2, p^2] = x[x,p^2] + [x,p^2]x = 2i\hbar(xp+px)$$

from previous result $[x,p^2] = -\hbar^2[x,\frac{d^2}{dx^2}] = 2i\hbar p$

also using $[x,p] = i\hbar \Rightarrow xp = px + i\hbar$ or $px = xp - i\hbar$

$$[x^2,p^2] = 2i\hbar(2xp - i\hbar) = 2i\hbar(2px + i\hbar)$$

(c) $F_n = [N,(a_+)^n]$

$n=1 \rightarrow F_1 = [N,a_+] = [a_+a_-, a_+] = a_+[a_-,a_+] + [a_+,a_+]a_-$

$$F_1 = a_+$$

$n=2 \rightarrow F_2 = [N,(a_+)^2] = a_+[N,a_+] + [N,a_+]a_+$

$$F_2 = (a_+)^2 + (a_+)^2 = 2(a_+)^2$$

Guess $F_n = [N,(a_+)^n] = n(a_+)^n$

assume it to be true for n, then

$$F_{n+1} = [N,(a_+)^{n+1}] = a_+[N,a_+^n] + [N,a_+]a_+^n$$

$$= a_+(n a_+^n) + (a_+(a_+^n)$$

$$F_{n+1} = (n+1)a_+^n$$

Thus by induction we proved that $F_n = [N,a_+^n] = n a_+^n$

$$H(a_+^n\phi_0) = \hbar\omega(N+\tfrac{1}{2})a_+^n\phi_0 = \tfrac{1}{2}\hbar\omega(a_+^n\phi_0) + \hbar\omega N(a_+^n\phi_0)$$

$$= \tfrac{1}{2}\hbar\omega(a_+^n\phi_0) + \{[N,a_+^n] + a_+^n N\}\hbar\omega\phi_0$$

$$= \tfrac{1}{2}\hbar\omega(a_+^n\phi_0) + \{n a_+^n + a_+^n 0\}\hbar\omega\phi_0$$

$$H(a_+^n\phi_0) = \hbar\omega(n+\tfrac{1}{2})a_+^n\phi_0 = E_n(a_+^n\phi_0)$$

but $H\phi_n = E_n\phi_n$ no degeneracy $\Rightarrow a_+^n\phi_0 = A_n\phi_n$

From your formula sheet you know that $a_+^n\phi_0 = \sqrt{n!}\,\phi_n$ i.e. $A_n = \sqrt{n!}$

Q3.

(a) $K = h\nu - W \geq 0 \Rightarrow \nu > \frac{W}{h} = \nu_c$

Thus photoelectric current will occur only if $\nu \geq \nu_c = \frac{W}{h}$

(b) If Intensity is Constant then $I = \frac{E}{tA}$ = Constant

But $E = n(h\nu)$; n = number of photon

$E = n(h\nu) = $ Cstant $\Rightarrow$ as $\nu \nearrow$ then $n \searrow$

Each photon gives one electron to photo current $\Rightarrow$ { Photo Current decreases as $n \searrow$

(C) $K = h\nu - W = \frac{hc}{\lambda} - W = e|V_s|$

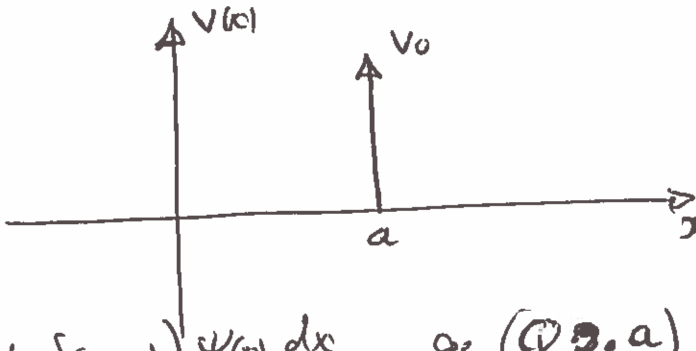
$$\Rightarrow |V_s| = \frac{hc}{\lambda e} - \frac{W}{e} = \frac{2\pi\hbar c}{\lambda e} - \frac{W}{e} = \frac{(6.28)(197.3\ eV.nm)}{400\ nm \times e} - 1.5V$$

$|V_s| = 3.1\ V - 1.5\ V = 1.6\ V$

(d) Interference experiment are the most important is exposing the wave like behavior. One such important in the History of Q.M. is Young double slit experiment for both electromagnetic and matter waves.

Q4.

$V(x) = V_0\, \delta(x-a)$



(a)

$$E = \int \psi^*(x)\, H\, \psi(x) = \int_{-\infty}^{+\infty} \psi^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\,\delta(x-a)\right)\psi(x)\, dx \quad \text{see } (Q2.a)$$

$$E = \frac{\hbar^2}{2m}\int \left|\frac{d\psi}{dx}\right|^2 dx + V_0\, |\psi(a)|^2 > 0$$

Since both quantities are definite positive.

(b) For $E > 0$ for both $x < a$ and $x > a$ we have

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) = -k^2\psi(x) \ ; \quad k = \frac{\sqrt{2mE}}{\hbar}$$

Thus

$$\psi(x) = \begin{cases} A e^{ik(x-a)} + B e^{-ik(x-a)} & \text{for } x < a \\ C e^{ik(x-a)} + D e^{-ik(x-a)} & \text{for } x > a \end{cases}$$

the use of $(x-a)$ is for convenience and ease of calculation, you can avoid it.

Let use $$\psi(x) = \begin{cases} A e^{ikx} + B e^{-ikx} & \text{for } x<a \\ C e^{ikx} + D e^{-ikx} & \text{for } x>a \end{cases}$$

then Schrodinger equation reads
$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) + \frac{2mV_0}{\hbar^2}\delta(x-a)\psi(x)$$

integrate this equation at the interface from $x=a-\varepsilon$ to $x=a+\varepsilon$
$$\int_{a-\varepsilon}^{a+\varepsilon} \frac{d^2\psi}{dx^2}\,dx = -\frac{2mE}{\hbar^2}\int_{a-\varepsilon}^{a+\varepsilon} dx\,\psi(x) + \frac{2mV_0}{\hbar^2}\int_{a-\varepsilon}^{a+\varepsilon}\delta(x-a)\psi(x)\,dx$$

$$\Rightarrow \left.\frac{d\psi}{dx}\right|_{a+\varepsilon} - \left.\frac{d\psi}{dx}\right|_{a-\varepsilon} \simeq -\frac{2mE}{\hbar^2}\psi(a)(2\varepsilon) + \frac{2mV_0}{\hbar^2}\psi(a)$$

limit as $\varepsilon \to 0$ give
$$\left.\frac{d\psi}{dx}\right|_{a^+} - \left.\frac{d\psi}{dx}\right|_{a^-} = \frac{2mV_0}{\hbar^2}\psi(a)$$

(C) using $\psi(a^+) = \psi(a^-)$; $\psi'(a^+) - \psi'(a^-) = \frac{2mV_0}{\hbar^2}\psi(a)$

gives
$$A e^{ika} + B e^{-ika} = C e^{ika} + D e^{-ika}$$
$$ik(Ce^{ika} - De^{-ika}) - ik(Ae^{ika} - Be^{-ika}) = \frac{2mV_0}{\hbar^2}(Ae^{ika} + Be^{-ika})$$

$$\Rightarrow \begin{cases} C e^{ika} + D e^{-ika} = A e^{ika} + B e^{-ika} \\ C e^{ika} - D e^{-ika} = A e^{ika}(1-2i\alpha) - B e^{-ika}(1+2i\alpha) \end{cases}$$
$$\alpha = \frac{mV_0}{\hbar^2 k}$$

$$\Rightarrow \begin{pmatrix} e^{ika} & e^{-ika} \\ e^{ika} & -e^{-ika} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} e^{ika} & e^{-ika} \\ e^{ika}(1-2i\alpha) & -e^{-ika}(1+2i\alpha) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

using $$\begin{pmatrix} e^{ika} & e^{-ika} \\ e^{ika} & -e^{-ika} \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} -e^{-ika} & -e^{-ika} \\ -e^{ika} & e^{ika} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-ika} & e^{-ika} \\ e^{ika} & -e^{ika} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-ika} & e^{-ika} \\ e^{ika} & -e^{ika} \end{pmatrix}\begin{pmatrix} e^{ika} & e^{-ika} \\ e^{ika}(1-2i\alpha) & -e^{-ika}(1+2i\alpha) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2-2i\alpha & e^{-2ika}(-2i\alpha) \\ e^{2ika}(2i\alpha) & 2+2i\alpha \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1-i\alpha & -i\alpha e^{-2ika} \\ i\alpha e^{2ika} & 1+i\alpha \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = M\begin{pmatrix} A \\ B \end{pmatrix}$$

$$M = \begin{pmatrix} 1-i\alpha & -i\alpha e^{-2ika} \\ i\alpha e^{2ika} & 1+i\alpha \end{pmatrix} ; \quad |M| = 1$$

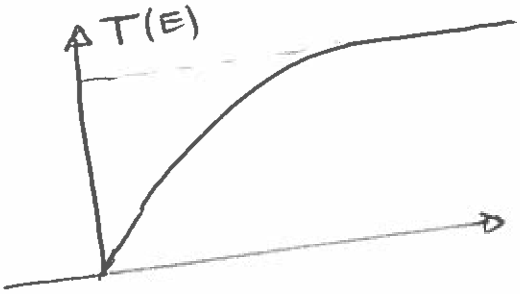
(d) $D = 0$ for an incident wave from far left, that

$$\begin{pmatrix} C \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\Rightarrow \begin{aligned} C &= M_{11}A + M_{12}B \\ 0 &= M_{21}A + M_{22}B \end{aligned} \quad \Rightarrow \quad B = -\frac{M_{21}}{M_{22}}A$$

then $C = A\left(M_{11} - \frac{M_{12}M_{21}}{M_{22}}\right) = A\frac{|M|}{M_{22}}$

$$\Rightarrow T = \left|\frac{C}{A}\right|^2 = \frac{|M|^2}{|M_{22}|^2} = \frac{1}{|M_{22}|^2} \quad \text{since } |M| = 1$$



$$T = \frac{1}{|M_{22}|^2} = \frac{1}{1+\alpha^2} = \frac{1}{1+\frac{(mV_0)^2}{\hbar^4 k^2}}$$

$$\alpha = \frac{mV_0}{\hbar^2 k} \quad ; \quad k^2 = \frac{2mE}{\hbar^2}$$

$$T(E) = \frac{1}{1+\frac{(mV_0)^2}{\hbar^2(2mE)}} = \frac{1}{1+\frac{E_0}{E}} = \frac{E}{E+E_0}$$

$$E_0 = \frac{mV_0^2}{2\hbar^2}$$