P2.5

A particle in the infinite square well has its initial wave function an even mixture of the first two stationary states:

$$\Psi(x,0) = A\left[\psi_1(x) + \psi_2(x)\right]$$

(a) Normalize $\Psi(x,0)$. (That is, find A. This is very easy, if you exploit the orthonormality of ψ_1 and ψ_2.)

(b) Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega \equiv \pi^2\hbar/2ma^2$.

(c) Compute $\langle x \rangle$. Notice that it oscillates in time. What is the angular freq. of the oscillation? What is the amplitude of the oscillation?

(d) Compute $\langle p \rangle$.

(e) If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of H. How does it compare with E_1 and E_2?

(a) $$\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1 \rightarrow \int_{-\infty}^{\infty} A^2(\psi_1^* + \psi_2^*)(\psi_1 + \psi_2)\, dx = A^2 \int_{-\infty}^{\infty} \psi_1^*\psi_1 + \cancelto{0}{\psi_1^*\psi_2} + \cancelto{0}{\psi_2^*\psi_1} + \psi_2^*\psi_2 \, dx$$

$$= A^2\left(\int_{-\infty}^{\infty} \psi_1^*\psi_1\, dx + \int_{-\infty}^{\infty} \psi_2^*\psi_2\, dx\right) = 1$$

$$V(x) = \begin{cases} 0, & 0 \le x \le a \\ \infty, & \text{elsewhere} \end{cases}$$

For infinite square well potential with length a: $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$

$$\Rightarrow \psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right), \quad \psi_2(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$$

$$\Rightarrow A^2\left(\frac{2}{a}\right)\left(\int_0^a \sin^2\left(\frac{\pi x}{a}\right)dx + \int_0^a \sin^2\left(\frac{2\pi x}{a}\right)dx\right) = 1$$

$$\sin^2\left(\frac{\pi x}{a}\right) = \frac{1-\cos\frac{2\pi x}{a}}{2}$$

$$\Rightarrow A^2\left(\frac{2}{a}\right)\left[\left(\frac{x}{2} - \cancel{\frac{a}{4\pi}\sin\frac{2\pi x}{a}}^{\,0}\right|_0^a + \left(\frac{x}{2} - \cancel{\frac{a}{8\pi}\sin\frac{4\pi x}{a}}^{\,0}\right|_0^a\right]$$

$$= A^2\left(\frac{2}{a}\right)\left[\frac{a}{2}+\frac{a}{2}\right] = 2A^2 = 1 \Rightarrow A = \frac{1}{\sqrt{2}}$$

(b) $\Psi(x,0) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a} + \sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\right) = \frac{1}{\sqrt{a}}\left(\sin\frac{\pi x}{a} + \sin\frac{2\pi x}{a}\right)$

$$\Psi(x,t) = A\left(\psi_1(x)e^{-i\frac{E_1}{\hbar}t} + \psi_2(x)e^{-i\frac{E_2}{\hbar}t}\right)$$

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2} = \omega\hbar, \quad E_2 = \frac{4\pi^2\hbar^2}{2ma^2} = 4\omega\hbar$$

$$\Rightarrow \Psi(x,t) = \frac{1}{\sqrt{a}}\left[\sin\left(\frac{\pi x}{a}\right)e^{-i\omega t} + \sin\left(\frac{2\pi x}{a}\right)e^{-i4\omega t}\right]$$

$$|\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t)$$

$$= \frac{1}{a}\left[\sin\left(\frac{\pi x}{a}\right)e^{+i\omega t} + \sin\left(\frac{2\pi x}{a}\right)e^{+i4\omega t}\right]\left[\sin\left(\frac{\pi x}{a}\right)e^{-i\omega t} + \sin\left(\frac{2\pi x}{a}\right)e^{-i4\omega t}\right]$$

$$= \frac{1}{a}\left[\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\left(e^{-i3\omega t} + e^{i3\omega t}\right) + \sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\right]$$

$$= \frac{1}{a}\left[2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t) + \sin^2\left(\frac{\pi x}{a}\right) + \sin^2\left(\frac{2\pi x}{a}\right)\right]$$

(c) $\langle x\rangle = \int_{-\infty}^{\infty} \Psi^*(x,t)\, x\, \Psi(x,t)\, dx$

$$= \frac{1}{a}\int_0^a x\left[\underbrace{2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)}_{I_1} + \underbrace{\sin^2\left(\frac{\pi x}{a}\right)}_{I_2} + \underbrace{\sin^2\left(\frac{2\pi x}{a}\right)}_{I_3}\right]$$

$$I_1: \int_0^a 2x \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)\,dx$$

$$= \cancel{2}\cos(3\omega t)\left[\frac{1}{\cancel{2}}\int_0^a x\left(\cos\frac{\pi x}{a} - \cos\frac{3\pi x}{a}\right)dx\right] \quad \text{IBP}$$

$f = x,\ g' = \cos\frac{3\pi x}{a}\,dx$

$f' = 1,\ g = \frac{a}{3\pi}\sin\frac{3\pi x}{a}$

$$= \cos(3\omega t)\left(\cancel{-\frac{ax}{3\pi}\sin\left(\frac{3\pi x}{a}\right)}^{0} - \frac{a^2}{9\pi^2}\cos\frac{3\pi x}{a} + \cancel{\frac{ax}{\pi}\sin\left(\frac{\pi x}{a}\right)}^{0} + \frac{a^2}{\pi^2}\cos\left(\frac{\pi x}{a}\right)\right)\Bigg|_0^a$$

$$= \frac{a^2}{\pi^2}\left(\left(\overset{-1}{\cos\pi} - \overset{1}{\cos 0}\right) - \frac{1}{9}\left(\overset{-1}{\cos 3\pi} - \overset{1}{\cos 0}\right)\right) = -\frac{16a^2}{9\pi^2}$$

$$I_2: \int_0^a x\sin^2\left(\frac{\pi x}{a}\right)dx = \left(\frac{x^2}{4} - \cancel{\frac{ax}{4\pi}\sin\left(\frac{2\pi x}{a}\right)}^{0} - \cancel{\frac{a^2}{8\pi^2}\cos\left(\frac{2\pi x}{a}\right)}^{-(1-1)=0}\right)\Bigg|_0^a = \frac{a^2}{4}$$

$$I_3: \int_0^a x\sin^2\left(\frac{3\pi x}{a}\right)dx = \left(\frac{x^2}{4} - \cancel{\frac{ax}{8\pi}\sin\left(\frac{4\pi x}{a}\right)}^{0} - \cancel{\frac{a^2}{32\pi^2}\cos\left(\frac{4\pi x}{a}\right)}^{-(1-1)=0}\right)\Bigg|_0^a = \frac{a^2}{4}$$

$$\Rightarrow \langle x\rangle = \frac{1}{\cancel{a}}\left(-\frac{16a^{\cancel{2}}}{9\pi^2}\cos(3\omega t) + \frac{a^{\cancel{2}}}{4} + \frac{a^{\cancel{2}}}{4}\right) = a\left(\frac{1}{2} - \frac{16}{9\pi^2}\cos(3\omega t)\right)$$

ang freq.

Angular frequency: $\omega' = 3\omega = \frac{3\pi^2\hbar}{2ma^2}$

Amplitude: $\frac{16a}{9\pi^2}$

(d) $\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \frac{16ma}{9\pi^2}\cdot 3\omega\sin(3\omega t) = \frac{16\cancel{m}\cancel{a}}{3\cancel{\pi^2}}\cdot\frac{\cancel{\pi^2}\hbar}{2\cancel{m}a^{\cancel{2}}}\sin(3\omega t)$

$$= \frac{8\hbar}{3a}\sin(3\omega t)$$

(e) We have two states: $n=1$ and $n=2$

$$\Rightarrow E_1 = \frac{\hbar^2 k_1^2}{2m} = \frac{\pi^2\hbar^2}{2ma^2}\ ;\ E_2 = \frac{2\pi^2\hbar^2}{ma^2}$$

"even mixture" of $\Psi_1, \Psi_2 \rightarrow P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2}$

$$\hat{H}\Psi = E\Psi$$

$$\langle H \rangle = \sum_{j=1}^{2} E_j \cdot P(E_j) = \underbrace{\frac{1}{2}(E_1 + E_2)}_{\frac{1}{N}\sum_{j}^{N} E_j \,=\, \text{average } E} = \frac{1}{2}\left(\frac{\pi^2\hbar^2}{2ma^2} + \frac{4\pi^2\hbar^2}{2ma^2}\right)$$

$$= \frac{5\pi^2\hbar^2}{2ma^2}$$

P2.7

A particle in the infinite square well has the initial wave function

$$\Psi(x,0) = \begin{cases} Ax, & 0 \le x \le \frac{a}{2}, \\ A(a-x), & \frac{a}{2} \le x \le a. \end{cases}$$

(a) Sketch $\Psi(x,0)$ and determine A.

(b) Find $\Psi(x,t)$.

(c) What is the probability that a measurement of the energy would yield the value E_1?

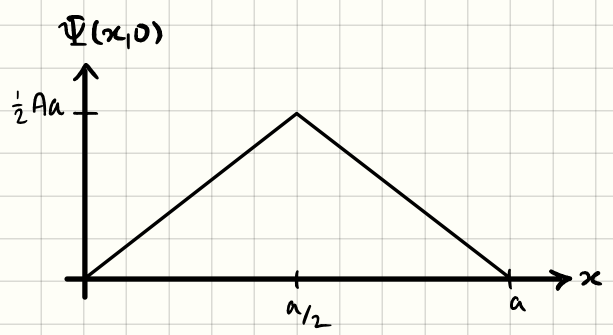
(d) Find the expectation value of the energy using $\langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n$

(a) $$\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = \int_0^{a/2} A^2 x^2 dx + \int_{a/2}^{a} A^2 (a-x)^2 dx = 1$$

$u = a - x$
$du = -dx$

$$\Rightarrow A^2 \left(\left(\frac{x^3}{3} \Big|_0^{a/2} + \left(-\frac{(a-x)^3}{3} \Big|_{a/2}^{a} \right) \right) = A^2 \left(\frac{a^3}{24} + \frac{a^3}{24} \right) = \frac{A^2 a^3}{12} = 1$$

$$\Rightarrow A = \sqrt{\frac{12}{a^3}}$$



(b) $$\Psi(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right) e^{-i\frac{E_n}{\hbar}t}, \quad 0 \le x \le a$$

$$\underbrace{\Psi(x,0)}_{f(x)} = \sum_{n=1}^{\infty} c_n \psi_n(x) = \sum_{n=1}^{\infty} c_n \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)}_{\psi_n(x)}$$

$$c_n = \int \psi_n^*(x) f(x)\,dx = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \Psi(x,0)\,dx$$

$$= \sqrt{\frac{2}{a}} A \left(\int_0^{a/2} x \sin\frac{n\pi x}{a}\,dx + \int_{a/2}^{a} (a-x) \sin\frac{n\pi x}{a}\,dx \right)$$

$f = x \quad dg = \sin\frac{n\pi x}{a}\,dx$

$df = dx \quad g = -\frac{a}{n\pi}\cos\frac{n\pi x}{a}$

$$= \sqrt{\frac{2}{a}} A \left(\left(\frac{a^2}{n^2\pi^2}\sin\frac{n\pi x}{a} - \cancel{\frac{ax}{n\pi}\cos\frac{n\pi x}{a}} \right|_0^{a/2} \right.$$

$\frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2}$

$$\left. + \left(-\frac{a^2}{n\pi}\cos\frac{n\pi x}{a} \right|_{a/2}^{a} + \left(-\frac{a^2}{n^2\pi^2}\sin\frac{n\pi x}{a} + \frac{ax}{n\pi}\cos\frac{n\pi x}{a} \right|_{a/2}^{a} \right)$$

$\cancel{-\frac{a^2}{n\pi}\cos n\pi}$

$+\frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{a^2}{n\pi}\cos n\pi}$

$$= \sqrt{\frac{24}{\cancel{a^4}}} \left(\frac{2\cancel{a^2}}{n^2\pi^2} \sin\frac{n\pi}{2} \right) = \frac{4\sqrt{6}}{n^2\pi^2}(-1)^{\frac{1}{2}(n-1)}, \quad n = 1, 3, 5, \ldots$$

$$\Psi(x,t) = \frac{8}{\pi^2}\sqrt{\frac{3}{a}} \sum_{\substack{n=1,\\ n\text{ odd}}}^{\infty} \frac{(-1)^{\frac{1}{2}(n-1)}}{n^2} \sin\left(\frac{n\pi x}{a}\right) e^{-i\frac{E_n}{\hbar}t}$$

(c) $|c_n|^2$ is the probability that a measurement of the energy would return the value E_n.

$$P(E_1) = |c_1|^2 = \left| \left(\frac{4\sqrt{6}}{\pi^2}\right) \underbrace{\cancel{(-1)^{\frac{1}{2}(1-1)}}}_{1} \right|^2 = \frac{16(6)}{\pi^4} = \frac{96}{\pi^4}$$

this is the case for all odd n subject to $||^2$

(d) $$\langle H \rangle = \sum_{n=1}^{\infty} |c_n|^2 E_n = \frac{96}{\cancel{\pi^4}_2} \sum_{\substack{n=1,\\ n\text{ odd}}}^{\infty} \frac{1}{n^{\cancel{4}2}} \cdot \frac{\cancel{n^2}\cancel{\pi^2}\hbar^2}{2ma^2} = \frac{48\hbar^2}{\pi^2 ma^2} \sum_{\substack{n=1,\\ n\text{ odd}}}^{\infty} \frac{1}{n^2} = \frac{48\hbar^2}{\cancel{\pi^2} ma^2} \cdot \frac{\cancel{\pi^2}}{8} = \frac{6\hbar^2}{ma^2}$$

P2.10

(a) Construct $\psi_2(x)$.
(b) Sketch ψ_0, ψ_1, and ψ_2.
(c) Check the orthogonality of ψ_0, ψ_1, and ψ_2 by explicit integration.
Hint: If you exploit the even-ness and odd-ness of the functions, there is really only one integral left to do.

(a) $\psi_2(x) = A_2 (\hat{a}_+)^2 \psi_0(x) = A_2 \hat{a}_+ \psi_1(x)$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_2(x) = \frac{A_2}{\sqrt{\cancel{2}\hbar \cancel{m\omega}}}\left(-\hbar\frac{d}{dx} + m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{\cancel{2m\omega}}{\hbar}}\, x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \underbrace{\frac{A_2}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}_{\gamma \equiv}\left(-\hbar\frac{d}{dx}\left(x e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right)$$

$$= \gamma\left[-\hbar\left(e^{-\frac{m\omega}{2\hbar}x^2} - \frac{m\omega}{\hbar}x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right) + m\omega x^2 e^{-\frac{m\omega}{2\hbar}x^2}\right]$$

① $\psi_2 = A_2\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)$

② $\int_{-\infty}^{\infty} \psi_2^* \psi_2 \, dx = 1 \rightarrow \int_{-\infty}^{\infty} A_2^2 \sqrt{\frac{m\omega}{\pi\hbar}}\, \underbrace{e^{-\frac{m\omega}{\hbar}x^2}}_{\text{even}} \left(\underbrace{\frac{2m\omega}{\hbar}x^2}_{\text{even}} \underbrace{-1}_{\text{even}}\right)^2 dx$

use $u = \sqrt{\frac{m\omega}{\hbar}}\, x \rightarrow du = \sqrt{\frac{\hbar}{m\omega}}\, dx$

$$\Rightarrow 2A_2^2 \sqrt{\frac{\cancel{m\omega}}{\pi \cancel{\hbar}}} \sqrt{\frac{\cancel{\hbar}}{\cancel{m\omega}}} \int_0^\infty (2u^2-1)^2 e^{-u^2} du = 1$$

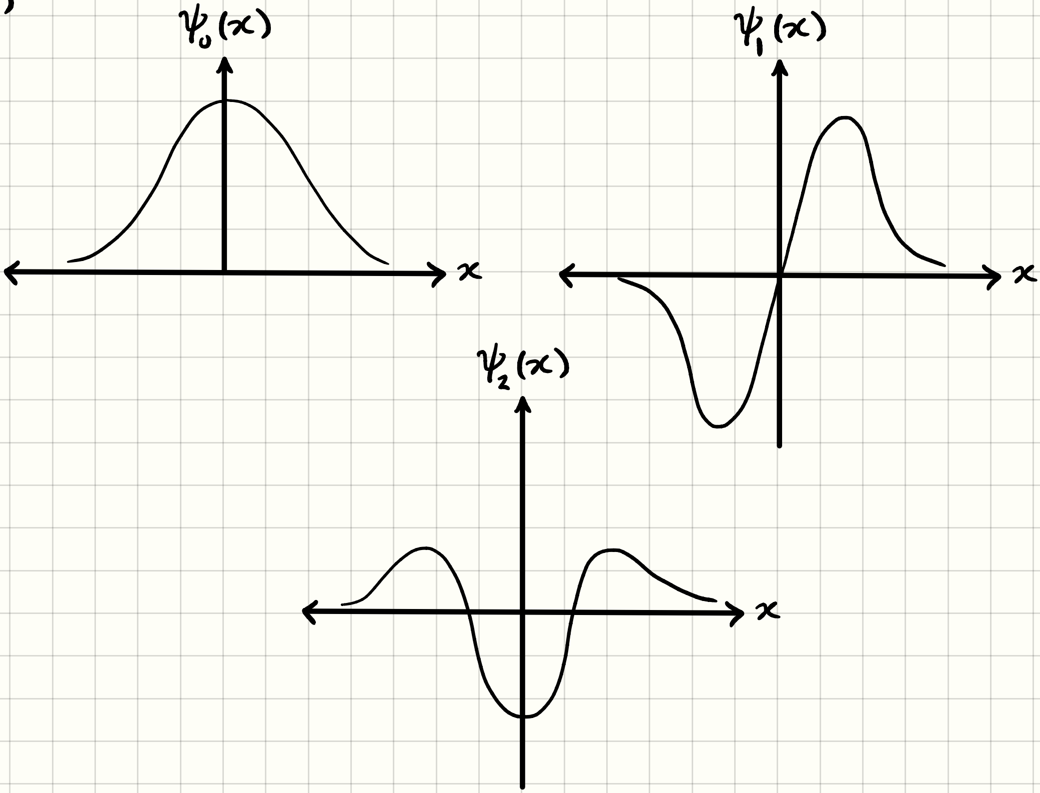
$$\Rightarrow 2A_2^2 \frac{1}{\sqrt{\pi}} \int_0^\infty (\underset{\substack{\uparrow \\ n=2}}{4u^4} - \underset{\substack{\uparrow \\ n=1}}{4u^2} + \underset{\substack{\uparrow \\ n=0}}{1}) e^{-u^2} du = 1; \quad \int_0^\infty x^{2n} e^{-\frac{x^2}{a^2}} dx = \sqrt{\pi}\, \frac{(2n)!}{(n)!} \left(\frac{a}{2}\right)^{2n+1}$$

$a=1$

$$\Rightarrow 2A_2^2 \underbrace{\left[\frac{48}{2^5} - \frac{8}{2^3} + \frac{1}{2}\right]}_{1} = 1 \rightarrow A_2 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \psi_2(x) = \frac{1}{\sqrt{2}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \left(\frac{2m\omega}{\hbar}x^2 - 1\right)$$

(b)



(c) $\boxed{\int_{-\infty}^{\infty} \psi_1^* \psi_0 \, dx = 0,}$ $\boxed{\int_{-\infty}^{\infty} \psi_1^* \psi_2 \, dx = 0}$

odd x even = odd

by symmetry ...

even x odd = odd

by symmetry ...

$$\boxed{\int_{-\infty}^{\infty} \psi_0^* \psi_2 \, dx} = \int_{-\infty}^{\infty} \sqrt{\frac{m\omega}{2\pi\hbar}} \left(\frac{2m\omega}{\hbar} x^2 - 1 \right) e^{-\frac{m\omega}{\hbar} x^2}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar}} \left(\int_{-\infty}^{\infty} \frac{2m\omega}{\hbar} x^2 e^{-\frac{m\omega}{\hbar} x^2} dx - \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar} x^2} dx \right)$$

$n=1$ $\quad$ $n=0$

$$= \sqrt{\frac{m\omega}{2\pi\hbar}} \left(\underbrace{\frac{2m\omega}{\hbar} \cdot \sqrt{\pi} \cdot \frac{2}{8} \sqrt{\frac{\hbar^3}{m^3\omega^3}} - \sqrt{\pi} \cdot \frac{1}{2} \sqrt{\frac{\hbar}{m\omega}}}_{0} \right)$$

$$\boxed{= 0}$$

P2.12

Find $\langle x \rangle$, $\langle p \rangle$, $\langle x^2 \rangle$, $\langle p^2 \rangle$, and $\langle T \rangle$, for the n^{th} stationary state of the harmonic oscillator, using the method of Example 2.5. Check that the uncertainty principle is satisfied.

From Example 2.5:

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-) \; ; \; \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right)$$

$$\hat{p}^2 = -\frac{\hbar m\omega}{2}\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right)$$

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_n^* \hat{x} \psi_n \, dx = \sqrt{\frac{\hbar}{2m\omega}} \int_{-\infty}^{\infty} \psi_n^* (\underbrace{a_+\psi_n}_{=\sqrt{n+1}\,\psi_{n+1}} + \underbrace{a_-\psi_n}_{=\sqrt{n}\,\psi_{n-1}}) \, dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\underbrace{\int_{-\infty}^{\infty}\psi_n^*\psi_{n+1}\,dx}_{0} + \sqrt{n}\underbrace{\int_{-\infty}^{\infty}\psi_n^*\psi_{n-1}\,dx}_{0}\right) = 0$$

$$\langle p \rangle = m\frac{d\langle x \rangle}{dt} = 0$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\underbrace{\int_{-\infty}^{\infty}\psi_n^*\psi_{n+2}\,dx}_{0} + n\int_{-\infty}^{\infty}\psi_n^*\psi_n\,dx + (n+1)\int_{-\infty}^{\infty}\psi_n^*\psi_n\,dx\right.$$
$$\left. + \sqrt{n(n-1)}\underbrace{\int_{-\infty}^{\infty}\psi_n^*\psi_{n-2}\,dx}_{0}\right) = (2n+1)\frac{\hbar}{2m\omega}$$

$$\langle p^2 \rangle = -\frac{\hbar m\omega}{2}\left(\sqrt{(n+1)(n+2)}\underbrace{\int_{-\infty}^{\infty}\psi_n^*\psi_{n+2}\,dx}_{0} - n\int_{-\infty}^{\infty}\psi_n^*\psi_n\,dx - (n+1)\int_{-\infty}^{\infty}\psi_n^*\psi_n\,dx\right.$$
$$\left. + \sqrt{n(n-1)}\underbrace{\int_{-\infty}^{\infty}\psi_n^*\psi_{n-2}\,dx}_{0}\right) = (2n+1)\frac{\hbar m\omega}{2}$$

$$\langle T\rangle = \frac{\langle p^2\rangle}{2m} = (2n+1)\frac{\hbar\omega}{4}$$

$$\sigma_x^2 = \langle x^2\rangle - \cancel{\langle x\rangle^2}^{\,0} = (2n+1)\frac{\hbar}{2m\omega}$$

$$\sigma_p^2 = \langle p^2\rangle - \cancel{\langle p\rangle^2}^{\,0} = (2n+1)\frac{\hbar m\omega}{2}$$

$$\Rightarrow \sigma_x\sigma_p = (2n+1)\sqrt{\frac{\hbar\cdot\hbar\cancel{m\omega}}{2\cancel{m\omega}\cdot 2}} = (2n+1)\frac{\hbar}{2} \geq \frac{\hbar}{2} \text{ for } n = 0,1,2,\ldots$$