P2.24

Check the uncertainty principle for the wave function in Eqn 2.132.
Hint: Calculating $\langle p^2 \rangle$ can be tricky, because the derivative of ψ has a step discontinuity at $x=0$. You may want to use the result in P2.23(b).
step function
Partial answer: $\langle p^2 \rangle = \left(\frac{m\alpha}{\hbar}\right)^2$

(Eqn 2.132) $\rightarrow \psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}, \quad E = -\frac{m\alpha^2}{2\hbar^2}$

$$\psi(x) = \begin{cases} \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha x/\hbar^2}, & x \geq 0 \\ \frac{\sqrt{m\alpha}}{\hbar} e^{+m\alpha x/\hbar^2}, & x < 0 \end{cases}$$

Uncertainty principle: $\sigma_x \sigma_p \geq \frac{\hbar}{2} \rightarrow$ need $\langle x \rangle, \langle x^2 \rangle, \langle p \rangle, \langle p^2 \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^* \hat{x} \psi \, dx = \int_{-\infty}^{\infty} x \underbrace{\frac{m\alpha}{\hbar^2} e^{-2m\alpha|x|/\hbar^2}}_{\text{even}} dx = 0$$

odd x even = odd

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^* \hat{x}^2 \psi \, dx = \int_{-\infty}^{\infty} x^2 \underbrace{\frac{m\alpha}{\hbar^2} e^{-2m\alpha|x|/\hbar^2}}_{\text{even}} dx = \frac{2m\alpha}{\hbar^2} \int_0^{\infty} x^2 e^{-2m\alpha x/\hbar^2} dx$$

even x even = even

$n=2, \; a = \frac{\hbar^2}{2m\alpha}$

$$= \frac{\cancel{2m\alpha}}{\cancel{\hbar}} (2)! \left(\frac{\hbar^2}{2m\alpha}\right)^{\cancel{3}\,2} = \frac{\hbar^4}{2m^2\alpha^2}$$

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = 0$$

$$\frac{\partial \psi}{\partial x} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left(-e^{-\frac{m\alpha x}{\hbar^2}} u(x) + e^{\frac{m\alpha x}{\hbar^2}} u(-x)\right)$$

$$= \begin{cases} e^{-\frac{m\alpha x}{\hbar^2}} & x > 0 \\ e^{\frac{m\alpha x}{\hbar^2}} & x < 0 \end{cases}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left(\frac{m\alpha}{\hbar^2} e^{-\frac{m\alpha x}{\hbar^2}} u(x) - e^{-\frac{m\alpha x}{\hbar^2}} \delta(x) + \frac{m\alpha}{\hbar^2} e^{\frac{m\alpha x}{\hbar}} u(-x) - e^{\frac{m\alpha x}{\hbar^2}} \delta(-x)\right)$$

$$= \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[\frac{m\alpha}{\hbar^2}\left(e^{-\frac{m\alpha|x|}{\hbar^2}}\right) - 2\delta(x)\right]$$

Note that $u(x) + u(-x) = 1$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \Psi^* \hat{p}^2 \Psi \, dx = -\hbar^2 \int_{-\infty}^{\infty} \psi^* \frac{\partial^2 \psi}{\partial x^2} dx$$

$$= -\hbar^2 \left(\frac{\sqrt{m\alpha}}{\hbar}\right)\left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \int_{-\infty}^{\infty} e^{-m\alpha|x|/\hbar^2} \left(\frac{m\alpha}{\hbar^2} e^{-m\alpha|x|/\hbar^2} - 2\delta(x)\right) dx$$

$$= \left(\frac{m\alpha}{\hbar}\right)^2 \left[2 - \frac{2m\alpha}{\hbar^2} \underbrace{\int_0^{\infty} e^{-\frac{2m\alpha x}{\hbar^2}} dx}_{-\frac{\hbar^2}{2m\alpha}(0-1)}\right] = \left(\frac{m\alpha\sqrt{2}}{\hbar}\right)^2 \left(1 - \frac{1}{2}\right)$$

$$= \left(\frac{m\alpha}{\hbar}\right)^2$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \cancel{\langle x\rangle^2}_{0}} = \frac{\hbar^2}{\sqrt{2}\, m\alpha}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \cancel{\langle p\rangle^2}_{0}} = \frac{m\alpha}{\hbar}$$

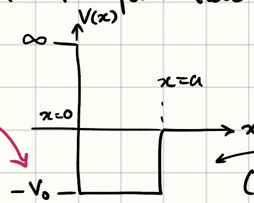
$$\sigma_x \sigma_p = \frac{\hbar^{\cancel{2}}}{\sqrt{2}\,\cancel{m\alpha}} \cdot \frac{\cancel{m\alpha}}{\cancel{\hbar}} = \frac{\hbar}{\sqrt{2}} = \frac{\hbar\sqrt{2}}{2} \geq \frac{\hbar}{2} \checkmark$$

P2.48

Consider a particle of mass m in the potential

piece-wise function

$$V(x) = \begin{cases} \infty, & x < 0 \\ -\dfrac{32\hbar^2}{ma^2}, & 0 \le x \le a \\ 0, & x > a \end{cases}$$



finite square well
(zero at origin ⇒ odd states)

(a) How many bound states are there? Bound state → $E < 0$

(b) In the highest-energy bound state, what is the probability that the particle would be found outside the well $(x > a)$?

Answer: 0.542, So even though it is "bound" by the well, it is more likely to be found outside!

(a) In the region $x > a$, the potential is zero:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \cancel{V(x)\psi} = E\psi \rightarrow \frac{d^2\psi}{dx^2} = K^2\psi \Rightarrow K \equiv \frac{\sqrt{-2mE}}{\hbar}, \; K > 0$$

$$\Rightarrow \psi(x) = Ae^{-Kx} \quad (x > a)$$

In the region $0 \le x \le a$, the potential is V_0:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \rightarrow \frac{d^2\psi}{dx^2} = -\ell^2\psi \Rightarrow \ell \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

$\psi(0) = 0 \Rightarrow D = 0$

$$\Rightarrow \psi(x) = C\sin(\ell x) + \cancel{D\cos(\ell x)} \quad (0 \le x \le a)$$

$$\Rightarrow \psi(x) = \begin{cases} Ae^{-Kx} & (x > a) \\ C\sin(\ell x) & (0 \le x \le a) \end{cases}$$

B.C. → $\psi_-(a) = \psi_+(a)$ — ①

$$\frac{d\psi_-}{dx}(a) = \frac{d\psi_+}{dx}(a) \text{ — ②}$$

Applying B.C.'s:

① $Ae^{-Ka} = C\sin(la)$

② $-KAe^{-Ka} = lC\cos(la)$

① in ②: $-K\cancel{C}\sin(la) = l\cancel{C}\cos(la)$

$-\frac{K}{l} = \cot(la)$ — (*)

$$K^2 + l^2 = -\frac{2mE}{\hbar^2} + \frac{2m}{\hbar^2}\left(\frac{32\hbar^2}{ma^2} + E\right)$$

$$= -\frac{2mE}{\hbar^2} + \frac{64}{a^2} + \frac{2mE}{\hbar^2}$$

$$= \frac{64}{a^2}$$

$$\Rightarrow K^2a^2 + l^2a^2 = 64$$

$$Ka = \sqrt{64 - l^2a^2}$$

$$-Ka = la\cot(la)$$

$$-\sqrt{64 - l^2a^2} = la\cot(la)$$

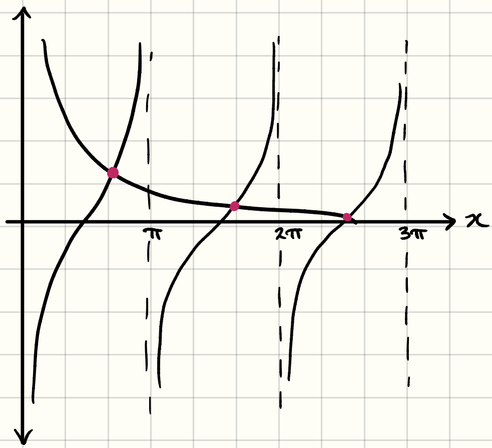
$$-\cot(la) = \sqrt{\frac{64}{l^2a^2} - 1}$$

Let $z \equiv la$, $z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}$

For $E = 0$, $-\cot(z) = \sqrt{\frac{z_0^2}{z^2} - 1}$

$\Rightarrow z_0 = 8 \rightarrow \frac{5\pi}{2} < z_0 < 3\pi$

three bound states



(b) $\int_{-\infty}^{\infty} |\psi|^2 dx = 1 = P(0 \leq x \leq a) + P(x > a)$

$$P_1 \equiv P(0 \leq x \leq a) = \int_0^a \psi^* \psi\, dx = |C|^2 \int_0^a \sin^2(lx)\, dx$$

$$= |C|^2 \int_0 \frac{1}{2} - \frac{1}{2}\cos(lx)\, dx = \frac{|C|^2}{2}\left(x - \frac{1}{2l}\sin(2lx)\right)\Big|_0^a$$

$$= \frac{|C|^2}{2}\left(a - \frac{1}{l}\sin(la)\cos(la)\right)$$

$$P_2 \equiv P(x > a) = \int_a^{\infty} \psi^* \psi\, dx = |A|^2 \int_a^{\infty} e^{-2Kx}\, dx = -\frac{|A|^2 e^{-2Kx}}{2K}\Big|_a^{\infty}$$

$$= \frac{|A|^2 e^{-2Ka}}{2K}$$

By continuity: $C\sin la = Ae^{-Ka}$

$$|A|^2 e^{-2Ka} = |C|^2 \sin^2(la)$$

$$\Rightarrow P_2 = \frac{|C|^2 \sin^2(la)}{2K}$$

$$P_1 + P_2 = \frac{|C|^2}{2}\left(a - \frac{1}{l}\sin(la)\cos(la) + \frac{1}{K}\sin^2(la)\right)$$

$$= \frac{|C|^2}{2K}\left(Ka - \frac{K}{l}\sin(la)\cos(la) + \sin^2(la)\right)$$

(*)

$$= \frac{|C|^2}{2K}\left(Ka + \frac{\cos(la)}{\cancel{\sin(la)}}\cancel{\sin(la)}\cos(la) + \sin^2(la)\right)$$

$$= \frac{|C|^2}{2K}\left(Ka + 1\right) = 1 \Rightarrow |C|^2 = \frac{2K}{Ka+1}$$

$$P(x > a) = P_2 = \frac{\cancel{2K}}{Ka+1} \cdot \frac{\sin^2(la)}{\cancel{2K}} = \frac{\sin^2(la)}{Ka+1}$$

$$\rightarrow Ka = \sqrt{z_0^2 - z^2} = -z\cot z \Rightarrow \cot^2 z = \frac{z_0^2}{z^2} - 1$$

$$\sin^2 z = \frac{1}{\csc^2 z} = \frac{1}{1+\cot^2 z} = \frac{z^2}{z_0^2} \longrightarrow P_2 = \frac{z^2/z_0^2}{1+\sqrt{z_0^2 - z^2}}$$

If $z_0 = 8$ and z is for third energy state (highest E), then z is the highest energy solution to $-\cot(z) = \sqrt{\frac{64}{z^2} - 1}$, which can be solved graphically to find $z = 7.95732$, giving

$$P_2 = \frac{z^2/z_0^2}{1+\sqrt{z_0^2 - z^2}} = 0.5420$$

P2.52

Consider the potential $V(x) = -\frac{\hbar^2 a^2}{m} \operatorname{sech}^2(ax)$ where a is a positive constant, and "sech" stands for the hyperbolic secant.

(a) ① Graph this potential. → Desmos

(b) ② Check that this potential has the ground state → TISE

$$\psi_0(x) = A\operatorname{sech}(ax)$$

and ③ find its energy. ④ Normalize ψ_0, and ⑤ sketch its graph.

(c) ⑥ Show that the function

$$\psi_k(x) = A\left(\frac{ik - a\tanh(ax)}{ik + a}\right)e^{ikx}$$

TISE ↑

(where $k \equiv \frac{\sqrt{2mE}}{\hbar}$, as usual) solves the Schrödinger equation for any (positive) energy E. Since $\tanh z \to -1$ as $z \to -\infty$,

no travel to the left!

$$\psi_k(x) \approx Ae^{ikx} \text{ for large negative } x,$$

This represents, then, a wave coming in from the left with <u>no accompanying reflected wave</u> (i.e. no term $\exp(-ikx)$).

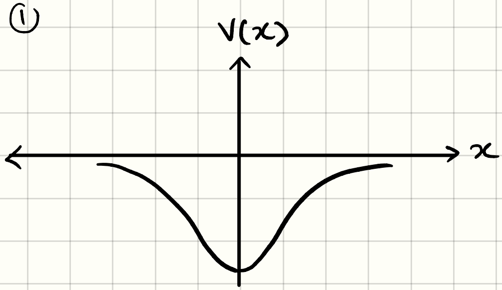
⑦ What is the asymptotic form of $\psi_k(x)$ at large <u>positive</u> x?

⑧ What are R and T for this potential?

Comment: This is a famous example of a <u>reflectionless potential</u>: every incident particle, regardless its energy, passes through.

(a)

①



(b) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi \quad , \quad \psi_0 = A\operatorname{sech}(ax)$

$$\Rightarrow -\frac{\hbar^2}{2m}\underbrace{\frac{d^2}{dx^2}\left(A\operatorname{sech}(ax)\right)} - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\left(A\operatorname{sech}(ax)\right) = E\psi$$

$$\frac{d}{dx}\left(-aA\tanh(ax)\operatorname{sech}(ax)\right) = a^2A\operatorname{sech}(ax)\left(\tanh^2(ax) - \operatorname{sech}^2(ax)\right)$$

$$= -\frac{\hbar^2 a^2}{2m}A\operatorname{sech}(ax)\left(\tanh^2(ax) - \operatorname{sech}^2(ax)\right) - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\left(A\operatorname{sech}(ax)\right)$$

$$= \cancel{-\frac{\hbar^2 a^2}{2m}A\operatorname{sech}^3(ax)} - \frac{\hbar^2 a^2}{2m}A\operatorname{sech}(ax)\left(1 - \cancel{\operatorname{sech}^2(ax)}\right)$$

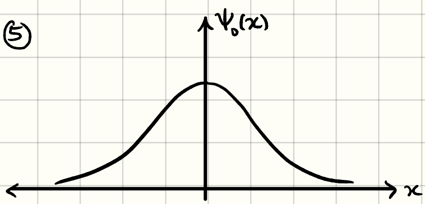
$$= -\frac{\hbar^2 a^2}{2m}A\operatorname{sech}(ax)$$

$$= -\frac{\hbar^2 a^2}{2m}\psi_0 = E\psi_0 \Rightarrow$$ ③ $E = -\frac{\hbar^2 a^2}{2m}$ and ② ψ_0 fulfills the TISE.

$$\int_{-\infty}^{\infty}\psi_0^*\psi_0\,dx = 1 \rightarrow |A|^2\int_{-\infty}^{\infty}\operatorname{sech}^2(ax)\,dx = \frac{|A|^2}{a}\tanh(ax)\Big|_{-\infty}^{\infty} = \frac{|A|^2}{a}\cdot 2 = 1$$

④ $\Rightarrow A = \sqrt{\frac{a}{2}}$

⑤



(c) $\psi_k(x) = A\left(\dfrac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx}$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_k(x) - \frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)\psi_k(x) = E\psi_k(x)$$

$$\underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_k(x)}_{}\;:\quad A\left(\frac{ik}{ik+a}\right)\frac{d^2}{dx^2}e^{ikx} - \frac{aA}{ik+a}\frac{d^2}{dx^2}\tanh(ax)e^{ikx}$$

$$= \frac{d}{dx}\frac{Ae^{ikx}}{ik+a}\left((ik)^2 - ika\tanh(ax) - a^2\operatorname{sech}^2(ax)\right)$$

$$= \frac{Ae^{ikx}}{ik+a}\left((ik)^3 - (ik)^2 a\tanh(ax) - ika^2\operatorname{sech}^2(ax) + 2a^3\tanh(ax)\operatorname{sech}^2(ax) - ika^2\operatorname{sech}^2(ax)\right)$$

$$= \frac{Ae^{ikx}}{ik+a}\left[+\frac{\hbar^2 ik}{2m}\left(+k^2 + ika\tanh(ax) + \cancel{2a^2\operatorname{sech}^2(ax)}\right) - \cancel{\frac{\hbar^2}{m}a^3\tanh ax\operatorname{sech}^2 ax}\right.$$
$$\left. - \cancel{\frac{\hbar^2 a^2}{m}\operatorname{sech}^2(ax)}\left(ik - \cancel{a\tanh(ax)}\right)\right]$$

$$= \frac{Ae^{ikx}\hbar^2}{2m(ik+a)}\left[ik^3 - k^2 a\tanh(ax)\right] = \frac{\hbar^2k^2}{2m}\underbrace{A\left(\frac{ik - a\tanh(ax)}{ik+a}\right)e^{ikx}}_{\psi_k(x)}$$

⑥

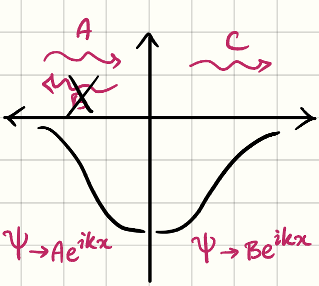
$$\Rightarrow \frac{\hbar^2k^2}{2m}\psi_k = E\psi_k \Rightarrow \psi_k \text{ fulfills TISE.}$$

$k = \dfrac{\sqrt{2mE}}{\hbar}$

⑦

$\tanh(ax) \to 1$ as $x \to +\infty \Rightarrow \psi_k(x) \to A\left(\dfrac{ik-a}{ik+a}\right)e^{ikx}$

no travel to the left!



There is no reflection:

⑧

$R = 0 \Rightarrow T = 1 - R = 1$