3.2

(a) For what range of ν is the function $f(x) = x^\nu$, in Hilbert space, on the interval $(0,1)$? Assume ν is real but not necessarily positive.

(b) For the specific case $\nu = 1/2$, is $f(x)$ in this Hilbert space? What about $xf(x)$? How about $\frac{d}{dx}(f(x))$?

(a) Want $f(x)$ such that $\int_a^b |f(x)|^2 dx < \infty$ $\to f(x) = x^\nu$, $(a,b) = (0,1)$, $\nu \in \mathbb{R}$

$$\Rightarrow \langle f|f\rangle = \int_0^1 x^{2\nu}\, dx = \frac{1}{2\nu+1} x^{2\nu+1}\Big|_0^1 = \frac{1}{2\nu+1}\left(1^{2\nu+1} - 0^{2\nu+1}\right)$$

this must not be (-)ve (else we have zero in the denominator)

if $0^0 \Rightarrow$ need to check it!

$$\Rightarrow 2\nu + 1 > 0 \to \nu > -\frac{1}{2}$$

$$\Rightarrow \nu = -\frac{1}{2} \to \langle f|f\rangle = \int_0^1 x^{-1}\, dx = \ln x\Big|_0^1 = \ln 1 - \ln 0 = +\infty \;\times \Rightarrow \text{does not converge}$$

$$\Rightarrow \boxed{\nu > -\frac{1}{2}}$$

(b) $$\int_0^1 x^{1/2} dx = \frac{1}{2(\frac{1}{2})+1}\left(\cancelto{1}{1^{2\frac{1}{2}+1}} - \cancelto{0}{0^{2\frac{1}{2}+1}}\right) = \frac{1}{2} < \infty \checkmark$$

$\Rightarrow$ $f(x)$ for $\nu = \frac{1}{2}$ is in Hilbert space.

$$\int_0^1 x \cdot x^{1/2} dx = \int_0^1 x^{3/2} dx = \frac{1}{2(\frac{3}{2})+1}(1-0) = \frac{1}{4} < \infty \checkmark$$

$\Rightarrow$ $xf(x)$ for $v = \frac{1}{2}$ is in Hilbert space.

$$\int_0^1 \frac{d}{dx}\left(x^{1/2}\right) dx = \int_0^1 \frac{1}{2} x^{-1/2} dx$$ does not converge, as shown before

$\Rightarrow$ $\frac{d}{dx}(f(x))$ for $v = \frac{1}{2}$ is NOT in Hilbert space.

3.7

(a) Suppose that $f(x)$ and $g(x)$ are two eigenfunctions of an operator $\hat{Q}$, with the same eigenvalue q. Show that any linear combination of f and g is itself an eigenfunction of $\hat{Q}$, with eigenvalue q.

(b) Check that $f(x) = e^{x}$ and $g(x) = e^{-x}$ are eigenfunctions of the operator d^2/dx^2, with the same eigenvalue. Construct two linear combinations of f and g that are orthogonal eigenfunctions on the interval $(-1, 1)$.

(a) $h(x) = af(x) + bg(x)$

We want to prove $\underline{\hat{Q}h = qh}$.

$\hat{Q}f = qf$ $\qquad$ $\hat{Q}g = qg$

$$\underline{\underline{\hat{Q}h}} = \hat{Q}(af + bg) = a\hat{Q}f + b\hat{Q}g = aqf + bqg = q(af+bg) = \underline{\underline{qh}} \checkmark$$

$\square$

(b) $\hat{Q} = \dfrac{d^2}{dx^2}$ $\quad\rightarrow\quad \hat{Q}f = \dfrac{d^2}{dx^2}e^{x} = e^{x} = f \Rightarrow q = 1$ $\quad\leftarrow$ same e'v

$\rightarrow\quad \hat{Q}g = \dfrac{d^2}{dx^2}e^{-x} = -\dfrac{d}{dx}e^{-x} = +e^{-x} = g \Rightarrow q = 1$

Try $h_1(x) = f + g = e^{x} + e^{-x}$, $h_2(x) = f - g = e^{x} - e^{-x}$

Are they orthogonal? $h_1(-x) = e^{-x} + e^{x} = h_1(x) \rightarrow$ even

$h_2(-x) = e^{-x} - e^{x} = -h_2(x) \rightarrow$ odd

$\Rightarrow$ Yes, they are orthogonal and well-defined for $x \in (-1, 1)$

3.13 Show that $\langle x \rangle = \int \Phi^* \left(i\hbar \frac{\partial}{\partial p} \right) \Phi \, dp$

Hint: Notice that $x e^{ipx/\hbar} = -i\hbar \frac{\partial}{\partial p} e^{ipx/\hbar}$, and use Eqn 2.147. In momentum space, then, the position operator is $i\hbar \frac{\partial}{\partial p}$. More generally,

$$\langle Q(x,p,t) \rangle = \begin{cases} \int \Psi^* \hat{Q}(x, -i\hbar \frac{\partial}{\partial x}, t) \Psi \, dx, & \text{in position space;} \\ \int \Phi^* \hat{Q}(i\hbar \frac{\partial}{\partial p}, p, t) \Phi \, dp, & \text{in momentum space.} \end{cases}$$

In principle, you can do all calculations in momentum space just as well (though not always as easily) as in position space.

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk \longrightarrow \text{eqn } 2.147$$

Let's use F.T. instead: $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \Phi(p,t) \, dp$

$$\langle x \rangle = \langle \Psi | \hat{x} \Psi \rangle = \frac{1}{2\pi\hbar} \int \left(\int_{-\infty}^{\infty} e^{-ipx/\hbar} \Phi^* dp \right) \underbrace{x}_{= -i\hbar \frac{\partial}{\partial p}} \left(\int_{-\infty}^{\infty} e^{ipx/\hbar} \Phi \, dp \right) dx$$

$$= \frac{1}{2\pi\hbar} \int \left(\int_{-\infty}^{\infty} e^{-ipx/\hbar} \Phi^* dp \right) \left(\int_{-\infty}^{\infty} -i\hbar \frac{\partial}{\partial p} (e^{ipx/\hbar}) \Phi \, dp \right)$$

$$\int_{-\infty}^{\infty} -i\hbar \frac{\partial}{\partial p} (e^{ipx/\hbar}) \Phi \, dp = \cancel{-i\hbar e^{ipx/\hbar} \Phi \Big|_{-\infty}^{\infty}} \, 0 \text{ (normalizability)} + \int_{-\infty}^{\infty} i\hbar e^{ipx/\hbar} \frac{\partial}{\partial p} \Phi \, dp$$

$$= \frac{1}{2\pi\hbar} \iiint e^{-i\bar{p}x/\hbar} \Phi^*(\bar{p}, t) \, i\hbar e^{ipx/\hbar} \frac{\partial}{\partial p} \Phi(p,t) \, dx \, d\bar{p} \, dp$$

$$\int e^{-i\bar{p}x/\hbar} e^{ipx/\hbar} dx = \int e^{i\overbrace{(p-\bar{p})}^{x}\overbrace{x/\hbar}^{k}} d\overbrace{x}^{k} \overset{\text{eqn 2.147!}}{=} 2\pi\hbar \, \delta(p - \bar{p})$$

$\delta(p-\bar{p})$ b/c δ-fn is even!

$$= \iint \Phi^*(\bar{p},t)\, i\hbar \frac{\partial}{\partial p} \Phi(p,t)\, \delta(\bar{p}-p)\, d\bar{p}\, dp$$

$$= \int \Phi^*(p,t)\, i\hbar \frac{\partial}{\partial p} \Phi(p,t)\, dp$$

$$= \int \Phi^* \left(i\hbar \frac{\partial}{\partial p} \right) \Phi \, dp \qquad \square$$

3.15 Prove the famous "(your name)" uncertainty principle," relating the uncertainty in position $(A = x)$ to the uncertainty in energy $(B = p^2/2m + V)$:

$$\sigma_x \sigma_H \geq \frac{\hbar}{2m} |\langle p \rangle|$$

For stationary states, this doesn't tell you much — why not?

$$[\hat{A}, \hat{B}] = [\hat{x}, \hat{H}] = \left[\hat{x}, \frac{\hat{p}^2}{2m} + V\right]$$

$$= \left[\hat{x}, \frac{\hat{p}^2}{2m}\right] + \cancelto{0}{[\hat{x}, V]}$$

$$= \frac{1}{2m}[\hat{x}, \hat{p}^2] \longrightarrow (\hat{x}\hat{p}^2 - \hat{p}^2\hat{x})$$

Operate $[\hat{x}, \hat{p}^2]$ on a function $f(x)$ with $\hat{p}^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$:

$$[\hat{x}, \hat{p}^2] f(x) = (\hat{x}\hat{p}^2 - \hat{p}^2\hat{x}) f(x)$$

$$= -\hbar^2 x \frac{d^2 f}{dx^2} + \hbar^2 \frac{d^2}{dx^2}(xf)$$

$$= -\hbar^2 x \frac{d^2 f}{dx^2} + \hbar^2 \frac{d}{dx}\left(f + x\frac{df}{dx}\right)$$

$$= \cancel{-\hbar^2 x \frac{d^2 f}{dx^2}} + \hbar^2 \frac{df}{dx} + \hbar^2 \frac{df}{dx} + \cancel{\hbar^2 x \frac{d^2 f}{dx^2}}$$

$$= 2\hbar^2 \frac{df}{dx} \Rightarrow \frac{1}{2m}[\hat{x}, \hat{p}^2] = \frac{\hbar^2}{m}\frac{\partial}{\partial x}$$

$$\sigma_x^2 \sigma_H^2 \geq \left(\frac{1}{2i}\left\langle \frac{\hbar^2}{m}\frac{\partial}{\partial x}\right\rangle\right)^2 = \left(\frac{\hbar}{2m}\left\langle \underbrace{\frac{\hbar}{i}\frac{\partial}{\partial x}}_{= -i\hbar\frac{\partial}{\partial x} = \hat{p}} \right\rangle\right)^2$$

from taking $\sqrt{\ }$

$$\Rightarrow \boxed{\sigma_x \sigma_H \geq \frac{\hbar}{2m}|\langle p \rangle|} \quad \square$$

3.20 Test the energy-time uncertainty principle for the wave function in Problem 2.5 and the observable x by calculating σ_H, σ_x, and $d\langle x\rangle/dt$ exactly.

The wave function in P2.5: $\Psi(x,t) = \frac{1}{\sqrt{a}}\left[\sin\left(\frac{\pi x}{a}\right)e^{-i\omega t} + \sin\left(\frac{2\pi x}{a}\right)e^{-i4\omega t}\right]$, $0 \le x \le a$

from HW2

$\omega = \frac{\pi^2\hbar}{2ma^2}$

$$= \frac{1}{\sqrt{2}}\psi_1 e^{-i\omega t} + \frac{1}{\sqrt{2}}\psi_2 e^{-i4\omega t}$$

"even mixture"

$$\Rightarrow P(E_1) = P(E_2) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

We want to show that $\Delta E \Delta t \geq \frac{\hbar}{2}$

$$\Delta E = \sigma_H \quad , \quad \sigma_H = \sqrt{\underset{②}{\langle H^2\rangle} - \underset{①}{\langle H\rangle}^2}$$

$$\Delta t = \frac{\sigma_x}{\left|\frac{d\langle x\rangle}{dt}\right|_{⑤}} \quad , \quad \sigma_x = \sqrt{\underset{④}{\langle x^2\rangle} - \underset{③}{\langle x\rangle}^2}$$

① $$\langle H\rangle = \sum_{n=1}^{2}|c_n|^2 E_n = P(E_1)\underbrace{E_1}_{\hbar\omega} + P(E_2)\underbrace{E_2}_{4\hbar\omega} = \frac{1}{2}\hbar\omega + 2\hbar\omega$$

$$= \frac{5}{2}\hbar\omega = \frac{5\pi^2\hbar^2}{4ma^2}$$

② $$\langle H^2\rangle = \sum_{n=1}^{2}|c_n|^2 E_n^2 = \frac{1}{2}(E_1^2 + E_2^2) = \frac{1}{2}(\hbar^2\omega^2 + 16\hbar^2\omega^2)$$

$$= \frac{17}{2}\hbar^2\omega^2 = \frac{17\pi^4\hbar^4}{8m^2a^4}$$

from HW2

③ $$\langle x\rangle = \langle\Psi|\hat{x}\Psi\rangle = a\left(\frac{1}{2} - \frac{16}{9\pi^2}\cos(3\omega t)\right)$$

$$\langle x^2\rangle = \langle \Psi | \hat{x}^2 \Psi \rangle = \int_{-\infty}^{\infty} \Psi^* x^2 \Psi \, dx$$

$$= \int_0^a x^2 \left[\frac{1}{\sqrt{a}} e^{+i\omega t} \sin\left(\frac{\pi x}{a}\right) + \frac{1}{\sqrt{a}} e^{+i4\omega t} \sin\left(\frac{2\pi x}{a}\right)\right]\left[\frac{1}{\sqrt{a}} e^{-i\omega t} \sin\left(\frac{\pi x}{a}\right) + \frac{1}{\sqrt{a}} e^{-i4\omega t} \sin\left(\frac{2\pi x}{a}\right)\right] dx$$

$$= \frac{1}{a} \int_0^a x^2 \Big[\underbrace{\sin^2\left(\frac{\pi x}{a}\right)}_{\boxed{I_1}} + \underbrace{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) e^{-i3\omega t} + \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) e^{i3\omega t}}_{\boxed{I_2}} + \underbrace{\sin^2\left(\frac{2\pi x}{a}\right)}_{\boxed{I_3}} \Big] dx$$

$$x^2 \quad = \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right)\left(e^{-i3\omega t} + e^{i3\omega t}\right)$$

$$= 2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right)\cos(3\omega t)$$

$$I_1: \int_0^a \frac{1}{2} x^2 \left(1 - \cos\frac{2\pi x}{a}\right) dx \overset{\text{IBP}}{=} \frac{1}{2}\left(\frac{a^3}{3} - \frac{a^3}{2\pi^2}\right)$$

prod.-to-sum formulas + IBP

$$I_2: 2\cos(3\omega t) \int_0^a x^2 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi x}{a}\right) dx = -\frac{16a^3}{9\pi^2}\cos(3\omega t)$$

$$I_3: \int_0^a \frac{1}{2} x^2 \left(1 - \cos\frac{4\pi x}{a}\right) dx \overset{\text{IBP}}{=} \frac{1}{2}\left(\frac{a^3}{3} - \frac{a^3}{8\pi^2}\right)$$

④ $$\langle x^2\rangle = \frac{1}{a}(I_1 + I_2 + I_3) = \frac{a^2}{6} - \frac{a^2}{4\pi^2} - \frac{16a^2}{9\pi^2}\cos(3\omega t) + \frac{a^2}{6} - \frac{a^2}{16\pi^2}$$

$$= a^2\left(\frac{1}{3} - \frac{5}{16\pi^2} - \frac{16}{9\pi^2}\cos 3\omega t\right)$$

$$\omega = \frac{\pi^2\hbar}{2ma^2}$$

$$= a^2\left(\frac{1}{3} - \frac{5}{16\pi^2} - \frac{16}{9\pi^2}\cos\frac{3\pi^2\hbar}{2ma^2}t\right)$$

⑤ $$\frac{d\langle x\rangle}{dt} = +\frac{16a}{3\pi^2}\omega \sin 3\omega t = \frac{8\hbar}{3ma}\sin\frac{3\pi^2\hbar}{2ma^2}t$$

$$\sigma_H = \sqrt{\langle H^2\rangle - \langle H\rangle} = \sqrt{\frac{17\pi^4\hbar^4}{8m^2a^4} - \frac{25\pi^4\hbar^4}{16m^2a^4}} = \frac{3\pi^2\hbar^2}{4ma^2}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$= \sqrt{a^2\left(\frac{1}{3} - \frac{5}{16\pi^2} - \cancel{\frac{16}{9\pi^2}\cos 3\omega t}\right) - a^2\left(\frac{1}{4} - \cancel{\frac{16}{9\pi^2}\cos 3\omega t} + \frac{256}{81\pi^4}\cos^2 3\omega t\right)}$$

$$= a\sqrt{\frac{1}{12} - \frac{5}{16\pi^2} - \frac{256}{81\pi^4}\cos^2 3\omega t} = \frac{a}{2}\sqrt{\frac{1}{3} - \frac{5}{4\pi^2} - \frac{1024}{81\pi^4}\cos 3\omega t}$$

$$\sigma_H\sigma_x = \frac{3\pi^2\hbar^2}{4ma^{\cancel{2}}}\cdot\frac{\cancel{a}}{2}\sqrt{\frac{1}{3} - \frac{5}{4\pi^2} - \frac{1024}{81\pi^4}\cos^2 3\omega t}$$

$$\frac{\sigma_H\sigma_x}{\left|\frac{d\langle x\rangle}{dt}\right|} = \frac{3\pi^2\hbar^2}{8ma}\cdot\sqrt{\frac{1/3 - 5/4\pi^2 - 1024/81\pi^4\cos^2 3\omega t}{\frac{16\hbar^2}{9m^2a^2}\sin^2 3\omega t}} = \Delta E\Delta t$$

$$= \frac{3\pi^2\hbar^{\cancel{2}}}{8\cancel{m}\cancel{a}}\cdot\frac{3\cancel{ma}}{4\cancel{\hbar}}\sqrt{\cdot} = \frac{9\pi^2\hbar}{32}\sqrt{\frac{1/3 - 5/4\pi^2 - \frac{1024}{81\pi^4}\cos^2 3\omega t}{\sin^2 3\omega t}}$$

Check if this is $\geq \frac{\hbar}{2}$?

$$\sigma_H\sigma_x \geq \frac{\hbar}{2}\left|\frac{d\langle x\rangle}{dt}\right|$$

just multiply both sides by $\left|\frac{d\langle x\rangle}{dt}\right|$

$$\Rightarrow \left(\frac{3\pi^2\hbar^2}{8ma}\right)^2\left(\frac{1}{3} - \frac{5}{4\pi^2} - \frac{1024}{81\pi^4}\cos^2 3\omega t\right) \geq \frac{\hbar^2}{4}\left(\frac{8\hbar}{3ma}\right)^2\sin^2 3\omega t$$

$$\Rightarrow \left(\frac{3a}{4}\cancel{\hbar\omega}\right)^2\left(\frac{1}{3} - \frac{5}{4\pi^2} - \frac{1024}{81\pi^4}\cos^2 3\omega t\right) \geq \left(\frac{8a}{3\pi^2}\cancel{\hbar\omega}\right)^2\sin^2 3\omega t$$

$$\frac{1}{3} - \frac{5}{4\pi^2} - \frac{1024}{81\pi^4}\cos^2 3\omega t \geqslant \left(\frac{32}{9\pi^2}\right)^2 \underbrace{\sin^2 3\omega t}$$

$$\left(\frac{32}{9\pi^2}\right)^2 = \frac{1024}{81\pi^4}$$

$$\underbrace{\frac{1}{3} - \frac{5}{4\pi^2}}_{0.2066819} \geqslant \underbrace{\left(\frac{32}{9\pi^2}\right)^2}_{0.1297823} \underbrace{\left(\cos^2 3\omega t + \sin^2 3\omega t\right)}_{1}$$

$$\boxed{0.2066819 \geqslant 0.1297823} \checkmark$$

The uncertainty principle is verified.