Solution

PHYS.310

QUIZ # 3 **Fall 212**

Q1. 1D Harmonic Oscillator

Given that

$$H=\hbar\omega(N+\frac{1}{2})\,,\ N=a_+a_-\,,\ [a_-,a_+]=1\,,\ N\Phi_n=n\Phi_n\,;\ n=0,1,2,3,\ldots;\ \int_{-\infty}^{+\infty}dx\,\Phi_n^*(x)\Phi_n(x)=\delta_{m,n}$$

(a) **Show** that $a_+\Phi_n$ is an **eigenstate** of H and **deduce** its associated **eigenenergy**.

$$H(a_+\phi_n)=([H,a_+]+a_+H)\phi_n=([N,a_+]\hbar\omega+a_+E_n)\phi_n=(E_n+\hbar\omega)(a_+\phi_n)$$

$\Rightarrow a_+\phi_n$ is eigenstate of H with eigenvalue $E_n+\hbar\omega = \boxed{E_{n+1}}$

(b) **Show** that $Na_-\Phi_n=(n-1)a_-\Phi_n$ and **deduce** the relationship $a_-\Phi_n=\sqrt{n}\,\Phi_{n-1}$

$$N(a_-\phi_n)=([N,a_-]+a_-N)\phi_n=a_-(N-1)\phi_n=a_-(n-1)\phi_n=(n-1)(a_-\phi_n)$$

Thus $a_-\phi_n$ is an eigenstate of N with eigenvalue $(n-1)$ since no degeneracy

$$\Rightarrow a_-\phi_n=A_n\phi_{n-1}\Rightarrow\int dx\,(a_-\phi_n)^*(a_-\phi_n)=\int dx\,A_n^*A_n\,\phi_{n-1}^*\phi_{n-1}$$

$$\Rightarrow\int dx\,\phi_n^*(a_+a_-\phi_n)=n\int dx\,\phi_n^*\phi_n=n=|A_n|^2\Rightarrow A_n=\sqrt{n}\ \text{or}\ \boxed{a_-\phi_{n-1}=\sqrt{n}\,\phi_{n-1}}$$

Q2. 1D Infinite Potential Well $V(x)=\begin{cases}0 & 0<x<a\\ \infty & x<0\,;\,x>a\end{cases}$

Given that Schrodinger eigenfunctions and associated eigenenergies are given by

$$\Psi_n(x)=\sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}x)\,;\ n=1,2,3,\ldots;\ E_n=\frac{\hbar^2\pi^2}{2ma^2}n^2;\ H\Psi_n(x)=E_n\Psi_n(x);\ \int_0^a dx\,\Psi_n^*(x)\Psi_m(x)=\delta_{m,n}$$

Consider the **wavefunction** $\Psi(x)=\frac{1}{\sqrt{3}}\Psi_2(x)+b\Psi_3(x)$, **b** being **a positive** number.

(a) **Find b** so that this wavefunction is **normalized** i.e. $\int_0^a dx\,\Psi^*(x)\Psi(x)=1$

$$\int_0^a\Psi^*(x)\Psi(x)\,dx=\int_0^a\left(\frac{1}{\sqrt3}\psi_2^*(x)+b^*\psi_3^*(x)\right)\left(\frac{1}{\sqrt3}\psi_2(x)+b\psi_3(x)\right)$$

$$=\frac{1}{3}\underbrace{\int_0^a\psi_2^*\psi_2\,dx}_{1}+\frac{b}{\sqrt3}\underbrace{\int\psi_2^*\psi_3\,dx}_{0}+\frac{b^*}{\sqrt3}\underbrace{\int\psi_3^*\psi_2\,dx}_{0}+|b|^2\underbrace{\int\psi_3^*(x)\psi_3(x)\,dx}_{1}$$

$$1=\frac{1}{3}+|b|^2\Rightarrow|b|^2=1-\frac{1}{3}=\frac{2}{3}\Rightarrow\boxed{b=\sqrt{\tfrac{2}{3}}\ \text{real positive number}}$$

(b) Calculate the **energy E** associated with this state, i.e. $E=\int_0^a dx\,\Psi^*(x)H\Psi(x)$

$$E=\int_0^a dx\left(\frac{1}{\sqrt3}\psi_2^*(x)+\sqrt{\frac{2}{3}}\psi_3^*(x)\right)H\left(\frac{1}{\sqrt3}\psi_2(x)+\sqrt{\frac{2}{3}}\psi_3(x)\right)dx$$

$$E=\frac{1}{3}\int_0^a\psi_2^*H\psi_2\,dx+\frac{\sqrt2}{3}\int\psi_2^*H\psi_3\,dx+\frac{\sqrt2}{3}\int\psi_3^*H\psi_2+\frac{2}{3}\int\psi_3^*(x)H\psi_3\,dx$$

$$E=\frac{1}{3}E_2+\frac{\sqrt2}{3}\left(E_3\underbrace{\int\psi_2^*\psi_3\,dx}_{0}+E_2\underbrace{\int\psi_3^*\psi_2\,dx}_{0}\right)+\frac{2}{3}E_3=\boxed{\frac{1}{3}(E_2+2E_3)}$$

$$E=\frac{1}{3}\left(\frac{\hbar^2\pi^2}{2ma^2}\right)(2^2+3^2\times2)=\boxed{\frac{22}{3}\left(\frac{\hbar^2\pi^2}{2ma^2}\right)}\ \cancel{=8\left(\frac{\hbar^2\pi^2}{2ma^2}\right)}=\frac{22}{3}\left(\frac{\hbar^2\pi^2}{2ma^2}\right)$$