Q1.

A (a) Using $|\Psi(t)\rangle = \sum_n C_n(t)\, e^{-iE_nt/\hbar}|\psi_n\rangle$; $H_0|\psi_n\rangle = E_n|\psi_n\rangle$

$H = H_0 + H'(t)$

$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = (H_0 + H'(t))|\Psi(t)\rangle$ gives $\omega_{mn} = (E_m - E_n)/\hbar$

$$i\hbar\, \dot{C}_m(t) = \sum_n e^{i\omega_{mn}t}\, C_n(t)\, H'_{mn}(t)$$

If initial conditions are such that $C_n(0) = \delta_{n,0}$ i.e. system in ground state $n=0$ at $t=0$, then to first order in H'

$$i\hbar\, \dot{C}_n(t) = e^{i\omega_{n0}t} H'_{n0}(t) \quad ; \quad \omega_{n0} = \frac{E_n - E_0}{\hbar} = n\omega$$

$$C_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{n0}t'} H'_{n0}(t')\, dt'$$

In particular Probability to find system in state $|n\rangle$ at $t=\infty$

$$P_{0n} = |C_n^{(1)}(\infty)|^2 = \left|\frac{1}{\hbar}\int_0^\infty e^{i\omega_{n0}t} H'_{n0}(t)\, dt\right|^2$$

$$P_{0n} = \left|\frac{1}{\hbar}\int_0^\infty e^{in\omega t}\, V_0 \langle n|x^4|0\rangle\, e^{-t/T}\right|^2$$

$$P_{0n} = \left(\frac{V_0}{\hbar}\right)^2 |\langle n|x^4|0\rangle|^2 \left|\int_0^\infty e^{t(-\frac{1}{T}+in\omega)}\right|^2$$

$$P_{0n} = \left(\frac{V_0}{\hbar}\right)^2 |\langle n|x^4|0\rangle|^2 \frac{1}{|\frac{1}{T} - in\omega|^2} = \left(\frac{V_0}{\hbar}\right)^2 \frac{T^2}{(n\omega T)^2+1}|\langle n|x^4|0\rangle|^2$$

$$\langle n|x^4|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \langle n|(a+a^\dagger)^4|0\rangle$$

$$= \left(\frac{\hbar}{2m\omega}\right)^2 \langle n|\left((a^\dagger)^4 + (a^\dagger)^2 a a^\dagger + a^\dagger a (a^\dagger)^2 + a(a^\dagger)^3 + a^2 a^{\dagger 2} + (a a^\dagger)^2\right)|0\rangle$$

$(a^\dagger)^4|0\rangle = \sqrt{1}(a^\dagger)^3|1\rangle = \sqrt{1.2}\,(a^\dagger)^2|2\rangle = \sqrt{1.2.3.4}\,|4\rangle = \sqrt{4!}\,|4\rangle$

$(a^\dagger)^3|0\rangle = \sqrt{3!}\,|0\rangle$ in general $(a^\dagger)^m|0\rangle = \sqrt{m!}\,|m\rangle$

$$\langle n|x^4|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \left\{\sqrt{4!}\,\delta_{n,4} + \left(\sqrt{2!} + (\sqrt{2})^3 + \sqrt{3!}\sqrt{3}\right)\delta_{n,2} + (2+1)\delta_{n,0}\right\}$$

$$\langle n|x^4|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^2 \left(\sqrt{4!}\,\delta_{n,4} + 6\sqrt{2}\,\delta_{n,2} + 3\,\delta_{n,0}\right)$$

$$P_{0n} = \left(\frac{V_0}{\hbar}\right)^2 \frac{T^2}{(n\omega T)^2+1}\left(\frac{\hbar}{2m\omega}\right)^4 \left(\sqrt{4!}\,\delta_{n,4} + 6\sqrt{2}\,\delta_{n,2} + 3\,\delta_{n,0}\right)^2$$

b) Possible transitions from the ground state are given by P_{04}, P_{02} and P_{00}

$$P_{00} = \frac{(V_0 T\hbar)^2}{(2m\omega)^4}\, 9 \quad ; \quad P_{0n} = \frac{(V_0 T\hbar)^2}{1+(n\omega T)^2}\frac{C_n}{(2m\omega)^4} \; ; \; \begin{cases} C_4 = 4! = 24 \\ C_2 = 72 \end{cases}$$

From the above result we just fix $n=0,2,4$ to get

$$P_0 = \frac{9(\hbar V_0 T)^2}{(2m\omega)^4} \quad ; \quad P_2 = \frac{72(\hbar V_0 T)^2}{(2m\omega)^4}\frac{1}{1+(2\omega T)^2}$$

$$P_4 = \frac{24(\hbar V_0 T)^2}{(2m\omega)^4}\frac{1}{1+(4\omega T)^2}$$

c/ Adiabatic limit $\omega T \gg 1$

$$P_2 = \frac{72(\hbar V_0 T)^2}{(2m\omega)^4}\frac{1}{4(\omega T)^2} = \frac{18(\hbar V_0 T)^2}{(2m\omega)^4(\omega T)^2}$$

$$P_4 = \frac{24(\hbar V_0 T)^2}{(2m\omega)^4}\frac{1}{16(\omega T)^2} = \frac{3}{2}\frac{(\hbar V_0 T)^2}{(2m\omega)^4(\omega T)^2}$$

So that $$\frac{P_0}{P_2} = \frac{9\cancel{(\hbar V_0 T)^2}}{\cancel{(2m\omega)^4}}\frac{\cancel{(2m\omega)^4}(\omega T)^2}{18\cancel{(\hbar V_0 T)^2}} = \frac{1}{2}(\omega T)^2$$

$$\frac{P_0}{P_4} = \frac{9\cancel{(\hbar V_0 T)^2}}{\cancel{(2m\omega)^4}}\cdot\frac{2\cancel{(2m\omega)^4}(\omega T)^2}{3\cancel{(\hbar V_0 T)^2}} = 6(\omega T)^2$$

$P_0 = \frac{1}{2}(\omega T)^2 P_2 \gg P_2$; $P_0 = 6(\omega T)^2 P_4 \gg P_4$

Thus in the adiabatic limit all transitions to $n\neq 0$ states are negligible so that the system prefers to remain the ofs ground state as dictated by the adiabatic approximation.

d) Non-adiabatic or Abrupt perturbation limit $\omega T \ll 1$

then $$P_2 \simeq \frac{72(\hbar V_0 T)^2}{(2m\omega)^4} \quad ; \quad P_4 \simeq \frac{24(\hbar V_0 T)^2}{(2m\omega)^4}$$

So that $$\frac{P_0}{P_2} = \frac{9}{72} = \frac{1}{8} \quad ; \quad \frac{P_0}{P_4} = \frac{9}{24} = \frac{3}{8}$$

or $$P_0 = \frac{1}{8}P_2 \quad ; \quad P_0 = \frac{3}{8}P_4$$ so that $P_2 > P_4 > P_0$

In this non-adiabatic limit the system prefers to make transitions to higher level states, contrary to the adiabatic limit.

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$$\frac{P_{00}}{P_{0n}} = \frac{9\,(V_0 T)^2}{(2m\omega)^2}\,\frac{(2m\omega)^2}{C_n (V_0 T)^2} = \frac{9}{C_n} = \begin{cases} \frac{9}{24} & \text{for } n=4 \\ \frac{9}{72} & \text{for } n=2 \end{cases}$$

$$P_{00} + P_{02} + P_{04} = 1 = P_{00}\left(1 + \frac{24}{9} + \frac{72}{9}\right) = P_{00}\left(\frac{35}{3}\right)$$

$$\Rightarrow P_{00} = \frac{3}{35} \; ; \; P_{02} = \frac{24}{35} \; ; \; P_{04} = \frac{8}{35}$$

So we see that that the system has $P_{02} \simeq 68.6\,\%$ to transit to $n=2$; $P_{04} \simeq 23\%$ to transit to $n=4$ and $P_{00} \simeq 8.4\,\%$ to remain in the ground state. A situation completely contrary to the adiabatic limit.

B. From a) we have

$$i\hbar\,\dot{C}_n(t) = \sum_m e^{i\omega_{nm} t}\, C_m(t)\, H'_{mn}(t)$$

The transition probability from state $|n\rangle$ at $t=0$ to state $|k\rangle$ at time t to first order in H'

$$C_m(0) = \delta_{m,n}$$

give

$$i\hbar\,\dot{C}_k(t) = e^{i\omega_{kn} t}\, H'_{nk}(t)$$

$$\Rightarrow \quad C_k^{(1)}(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_{kn} t'}\, H_{kn}(t')\, dt'$$

$$P_{nk}(t) = |C_k^{(1)}(t)|^2 = \frac{1}{\hbar^2}\left|\int_0^t e^{i\omega_{nk} t'}\, H_{nk}(t')\, dt'\right|^2$$

$$\omega_{nk} = \frac{E_k - E_n}{\hbar} \; ; \; \omega_{kn} = \frac{E_n - E_k}{\hbar}$$

$$P_{kn}(t) = |C_n^{(1)}(t)|^2 = \frac{1}{\hbar^2}\left|\int_0^t e^{i\omega_{kn} t'}\, H_{kn}(t')\, dt'\right|^2$$

using the fact that:

$$\omega_{kn} = -\omega_{nk} \; ; \; H_{kn}(t) = H^*_{nk}(t)$$

then

$$P_{kn}(t) = \frac{1}{\hbar^2}\left|\int_0^t e^{-i\omega_{nk} t'}\, H^*_{nk}(t')\, dt'\right|^2$$

$$= \frac{1}{\hbar}\left|\left(\int_0^t e^{i\omega_{nk} t'}\, H_{nk}(t')\, dt'\right)^*\right|^2 \quad \text{since } |z| = |z^*|$$

$$P_{kn}(t) = \frac{1}{\hbar}\left|\int_0^t e^{i\omega_{nk} t'}\, H_{nk}(t')\, dt'\right|^2 = P_{nk}(t)$$

If we assume that $E_k > E_n$ then $\omega_{nk} > 0$ and P_{nk} correspond to absorption rate while P_{kn} correspond to stimulated emission, the main important process in atoms interacting with an electromagnetic field.

Q2.

A. $\psi^{WKB}_{(x)}$ should be identical in $x_1 < x < x_2$ region, let us write

$$\psi^{WKB}(x) = \begin{cases} \dfrac{2C_1}{\sqrt{p(x)}} \sin\theta_1 & \text{for } x > x_1 \\ \dfrac{2C_2}{\sqrt{p(x)}} \sin\theta_2 & \text{for } x < x_2 \end{cases}$$

$$\theta_2 = \frac{1}{\hbar}\int_x^{x_2} p(x')dx' + \frac{\pi}{4}$$

$$\theta_1 = \frac{1}{\hbar}\int_{x_1}^{x} p(x')dx' + \frac{\pi}{4} = \frac{1}{\hbar}\int_{x_1}^{x_2} p(x')\,dx' + \frac{1}{\hbar}\int_{x_2}^{x} p(x')\,dx' + \frac{\pi}{4}$$

$$\theta_1 = \frac{1}{\hbar}\int_{x_1}^{x_2} p(x')dx' + \frac{\pi}{2} - \theta_2 = \delta - \theta_2 \;; \quad \delta = \frac{1}{\hbar}\int_{x_1}^{x_2} p(x)\,dx + \frac{\pi}{2}$$

$$\sin\theta_1 = \sin(\delta - \theta_2) = \sin\delta\cos\theta_2 - \cos\delta\sin\theta_2$$

imposing that $\dfrac{2C_1}{\sqrt{p(x)}}\sin\theta_1 = \dfrac{2C_2}{\sqrt{p(x)}}\sin\theta_2$

gives

$$\frac{2C_1}{\sqrt{p(x)}}\left(\sin\delta\cos\theta_2 - \cos\delta\sin\theta_2\right) = \frac{2C_2}{\sqrt{p(x)}}\sin\theta_2$$

$\Rightarrow$ require $\sin\delta = 0$ (no $\cos\theta_2$ term)

$$\delta = \frac{1}{\hbar}\int_{x_1}^{x_2} p(x)\,dx + \frac{\pi}{2} = n\pi \;; \quad n = 1,2,3\ldots$$

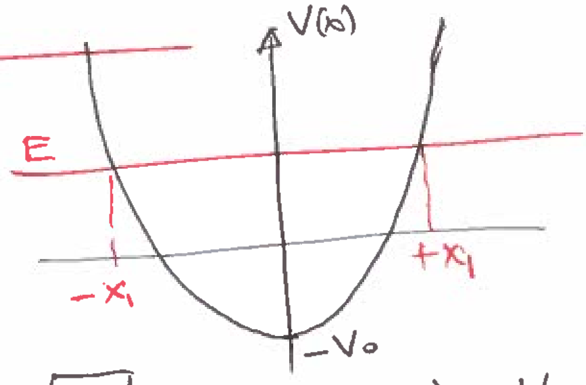
$$\Rightarrow \frac{1}{\hbar}\int_{x_1}^{x_2} p(x)\,dx = \left(n - \frac{1}{2}\right)\pi \;; \quad n = 1,2,3,\ldots$$

or

$$\boxed{\frac{1}{\hbar}\int_{x_1}^{x_2} p(x)\,dx = \left(n' + \frac{1}{2}\right)\pi \;; \quad n' = n-1 = 0,1,2,\ldots}$$

B.

a) Turning points are given by



$$E = V_0\left(\frac{x^2}{a^2} - 1\right)$$

$$\Rightarrow \frac{x^2}{a^2} = \left(\frac{E}{V_0} + 1\right)$$

$$\Rightarrow x = \pm a\sqrt{\frac{E}{V_0} + 1} = \pm x_1 \;; \quad x_1 = a\sqrt{\frac{E}{V_0}+1} > 0 \quad \text{i.e. } E > -V_0$$

Bound state are given by

$$\int_{x_1}^{+x_1}\sqrt{2m(E-V(x))}\,dx = (n+\tfrac{1}{2})\pi\hbar \quad ; \quad x_1 = a\sqrt{\frac{E}{V_0}+1} \qquad 5$$

$$\int_{-x_1}^{+x_1}\sqrt{2m\left(E-V_0\left(\frac{x^2}{a^2}-1\right)\right)}\,dx = 2\int_0^{x_1}\sqrt{2m\left(E+V_0-V_0\frac{x^2}{a^2}\right)}\,dx$$

$$= 2\sqrt{2mV_0}\int_0^{x_1}\sqrt{\frac{E}{V_0}+1-\frac{x^2}{a^2}}\,dx = \frac{2\sqrt{2mV_0}}{a}\int_0^{x_1}\sqrt{x_1^2-x^2}\,dx$$

$$= \frac{2\sqrt{2mV_0}}{a}x_1^2\int_0^{x_1}\sqrt{1-\left(\frac{x}{x_1}\right)^2}\,\frac{dx}{x_1} = \frac{2\sqrt{2mV_0}}{a}x_1^2\int_0^1\sqrt{1-u^2}\,du$$

$$= \frac{2\sqrt{2mV_0}}{a}x_1^2\frac{1}{2}\sin^{-1}u\Big|_0^1 = \frac{\pi\sqrt{2mV_0}}{2a}x_1^2 = \frac{\pi\sqrt{2mV_0}}{2a}\left(\frac{E}{V_0}+1\right)a^2$$

$$\Rightarrow \quad \frac{\cancel{\pi}a\sqrt{2mV_0}}{2}\left(\frac{E}{V_0}+1\right) = (n+\tfrac{1}{2})\cancel{\pi}\hbar$$

$$\Rightarrow \quad E_n = V_0\left[(n+\tfrac{1}{2})\frac{2\hbar}{a\sqrt{V_0}\sqrt{2m}}-1\right]$$

$$\boxed{E_n = (n+\tfrac{1}{2})\hbar\left(\frac{\sqrt{2V_0}}{\sqrt{m}\,a}\right)-V_0} \qquad n=0,1,2,\ldots$$

(b) which is the expected result if we compare $V(x) = \frac{V_0}{a^2}x^2 - V_0$ and write it as $V(x) = \frac{1}{2}m\omega^2x^2 - V_0$

$$\Rightarrow \quad \frac{1}{2}m\omega^2 = \frac{V_0}{a^2} \quad \Rightarrow \quad \omega = \left(\frac{2V_0}{ma^2}\right)^{1/2}$$

Also note/ The ground state energy obtains for $n=0$

$$E_0 = \frac{\hbar\sqrt{V_0}}{a\sqrt{2m}} - V_0 < 0 \Rightarrow \frac{\hbar\sqrt{V_0}}{a\sqrt{2m}} < V_0$$

$$\Rightarrow \sqrt{V_0} > \frac{\hbar}{a\sqrt{2m}} \Rightarrow V_0 > \frac{\hbar^2}{2ma^2}$$

The Hamiltonian of our system is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\left(\frac{x^2}{a^2}-1\right) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2 - V_0$$

So that $\quad E_n = \hbar\omega(n+\tfrac{1}{2}) - V_0$

in our case identification gives: $\frac{1}{2}m\omega^2 = \frac{V_0}{a^2} \Rightarrow \omega = \sqrt{\frac{2V_0}{ma^2}}$

So that: $\quad E_n = (n+\tfrac{1}{2})\hbar\sqrt{\frac{2V_0}{ma^2}} - V_0$

which is the result obtained above in (a).

c/ From the bound state formula

$$E_n = (n+\tfrac{1}{2})\hbar\sqrt{\frac{2V_0}{ma^2}} - V_0$$

we see that $E_0 < E_1 < E_2 < \cdots < E_N < E_{N+1} \cdots$

So if we want (N+1) bound states with negative energy, then we require

$$E_N = (N+\tfrac{1}{2})\hbar\sqrt{\frac{2V_0}{ma^2}} - V_0 < 0$$

$$\Rightarrow \quad (N+\tfrac{1}{2})\hbar\sqrt{\frac{2}{ma^2}} < \sqrt{V_0}$$

$$\Rightarrow \quad \begin{cases} V_0 > (2N+1)^2 \dfrac{\hbar^2}{2ma^2} = (2N+1)^2 \epsilon_0 \\ \epsilon_0 = \dfrac{\hbar^2}{2ma^2} \end{cases}$$

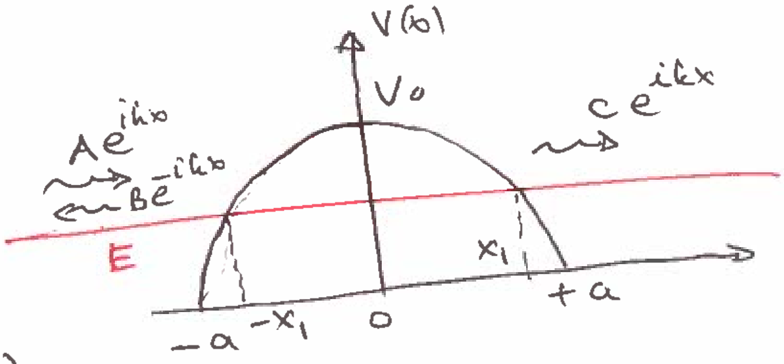
So that $N=1$ bound state $E_0 < 0 \Rightarrow V_0 > \epsilon_0$

$N=2$ bound state $E_0, E_1 < 0 \Rightarrow V_0 > 9\epsilon_0$

Thus if you compute bound states as a function of V_0 you will see negative energy bound state pop whenever you cross values of V_0 equal to: $\epsilon_0, 9\epsilon_0, 25\epsilon_0, \cdots (2N+1)^2\epsilon_0$.

C.

The particle is incident on a barrier so it will reflect partly and transmit partially.



$$T = \left|\frac{C}{A}\right|^2 = \exp(-2\gamma)$$

$$\gamma = \frac{1}{\hbar}\int_{-x_1}^{x_1} dx\sqrt{2m(V(x)-E)} = \frac{1}{\hbar}\int_{-x_1}^{+x_1} dx\sqrt{2m\left(V_0 - E - \frac{V_0}{a^2}x^2\right)}$$

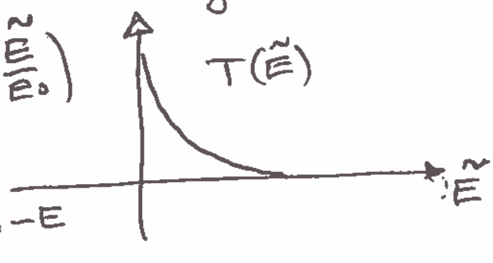
turning points at $E = V(x) = V_0\left(1-\frac{x^2}{a^2}\right)$

$$\Rightarrow \quad \frac{x^2}{a^2} = 1 - \frac{E}{V_0} \Rightarrow x = \pm a\sqrt{1-\frac{E}{V_0}} = \pm x_1$$

$$\gamma = \frac{2}{\hbar}\int_0^{x_1} dx\sqrt{\frac{2mV_0}{a^2}(x_1^2 - x^2)} = \frac{2}{\hbar}\frac{\sqrt{2mV_0}}{a}x_1^2\int_0^1\sqrt{1-u^2}\,du$$

$$\gamma = \frac{\pi}{2}a\sqrt{2mV_0}\left(1-\frac{E}{V_0}\right) \Rightarrow T = T_0\exp\left(-\frac{\tilde{E}}{E_0}\right)$$

$$T_0 = \exp\left(\frac{\pi a}{2}\sqrt{2mV_0}\right); \quad E_0 = \frac{1}{\pi a}\sqrt{\frac{2V_0}{m}}$$



$\tilde{E} = V_0 - E$