PHYS.410

QUIZ # 6 **Fall 221**

Q.

Consider the scattering of particles by a **central potential** V(r) and start from the integral form of Schrodinger equation **given by**

($\Psi^0(r) = e^{ikz}$ is the incident plane wave, **zeroth order** approximation)

$$\Psi(r) = \Psi^0(r) + \int G(r-r')U(r')\Psi(r')d^3r' \quad ; \quad V(r) = \frac{\hbar^2}{2m}U(r) \quad ; \quad G(r-r') = -\frac{1}{4\pi}\frac{e^{ik|r-r'|}}{|r-r'|}$$

a) **Show that** the **n-th order** Born approximation, **denoted** $\Psi^n(r)$, can be written in the following compact form (the superscript **n** is the approximation order not a power!)

$$\Psi^n(r) = \Psi^0(r) + \int G(r-r')U(r')\Psi^{n-1}(r')d^3r' \qquad \textbf{(5Pts.)}$$

where $\Psi^{n-1}(r)$ is the (n-1) Born approximation for $\Psi(r)$. (**Do not** use explicit form of G)

b) Compute the **scattering amplitude**, $f(\theta)$, **to the first order** Born approximation for the 3D **square well** potential defined by

$$V(r) = \begin{cases} -V_0 & r \le a \\ 0 & r > a \end{cases}$$

and **show that the scattering amplitude** $f(\theta) = -\frac{2m}{\hbar^2 K}\int_0^\infty rV(r)\sin(Kr)dr$ becomes

$$f(\theta) = \frac{2mV_0}{\hbar^2 K^3}\left[\sin Ka - Ka\cos Ka\right] \quad ; \quad K = 2k\sin(\theta/2) \quad ; \quad E = \frac{(\hbar k)^2}{2m} \qquad \textbf{(7Pts.)}$$

Find $f(\theta)$ in the limit Ka << 1. (**use** $\sin x = x - \frac{1}{6}x^3$; $\cos x = 1 - \frac{1}{2}x^2$; $x \ll 1$) **(5Pts.)**

Deduce that the total cross section is given by $\sigma = \int |f(\theta)|^2 d\Omega = 4\pi \left(\frac{2mV_o a^3}{3\hbar^2} \right)^2$ **(3Pts.)**

Solution Quiz # 6

a/ Using our integral equation we have

$n=1$ $\quad \Psi^1(n) = \Psi^0(n) + \int G(n-n')\, U(n')\, \Psi^0(n')\, d^3n'$ — to first order in U

$n=2$ $\quad \Psi^2(n) = \Psi^0(n) + \int G(n-n')\, U(n')\, \Psi^1(n')\, d^3n'$ — to second order in U

$$\Psi^n(n) = \Psi^0(n) + \int G(n-n')\, U(n')\, \Psi^{n-1}(n')\, d^3n'$$

thus $\quad \Psi^n(n) = \Psi^0(n) + \int G(n-n')\, U(n') \Psi^0(n'))$

$$\Psi^2(n) = \Psi^0(n) + \int G(n-n')\, U(n')\, d^3n' \left(\Psi^0(n') + \int d^3n''\, G(n'-n'')\, U(n'')\, \Psi^0(n'')\right)$$

$$= \Psi^0(n) + \int G(n-n')\, U(n')\, d^3n'\, \Psi^0(n') + \int G(n-n')\, U(n')\, d^3n' \int d^3n''\, G(n'-n'')\, U(n'')\, \Psi^0(n'')$$

$$= \Psi^0(n) + \int G(n-n')\, U(n')\, \Psi^0(n')\, d^3n' + \int G(n-n')\, U(n')\, d^3n' \int d^3n''\, G(n'-n'')\, U(n'')\, \Psi^0(n'')$$

$$\Psi(n) = \Psi^0 + \int G U\, d^3n_1 \int G U\, d^3n_2 \int G U\, d^3n_3\, \Psi^0 + \cdots$$

$$+ \int G U\, d^3n_1 \int G U\, d^3n_2 \int G U\, d^3n_3\, \Psi^0$$

$$\Psi^{n+1}(n) = \Psi^0(n) + \int G(n-n')\, U(n')\, d^3n' \Big(\Psi^0(n') + \underbrace{\int G(n'-n'')\, U(n'')\, \Psi^{n-1}(n'')\, d^3n''}_{\Psi^n(n'')}\Big)$$

$$= \Psi^0(n) + \int G(n-n')\, U(n')\, \Psi^0(n')\, d^3n'$$

$$+ \int G(n-n')\, U(n')\, d^3n' \int G(n'-n'')\, U(n'')\, \Psi^{n-1}(n'')\, d^3n''$$

$$+ \int G(n-n')\, U(n')\, d^3n' \int G U\, d^3n_2\, \Psi^0 + \int G U\, d^3n_1 \int G U\, d^3n_2 \int G U\, d^3n_3\, \Psi^0$$

Symbolically

$$\Psi(n) = \Psi^0(n) + \int G U \Psi^0\, d^3n_1 + \int G U\, d^3n_1 \int G U\, d^3n_2\, \Psi^0 + \cdots$$

$$+ \left(\int G U\, dn\right)^n \Psi_0 + \cdots$$

$$\Psi^1(n) = \Psi^0 + \int G U \Psi^0\, d^3n_1$$

$$\Psi^2(n) = \Psi^0 + \int G U\, d^3n_1 \int G U \Psi^0\, d^3n_2 + \int G U \Psi^0\, d^3n_1$$

$$= \Psi^0 + \int G U\, d^3n_1 \left(\Psi^0 + \int G U \Psi^0\, d^3n_2\right)$$

$$= \Psi^0 + \int G U \Psi^1\, d^3n_1$$

$$\Psi^2 = \Psi^0 + \int G U \Psi^0\, d^3n_1 + \int G U\, d^3n_1 \int G U \Psi^0\, d^3n_2 + \cdots + \left(\int G U\, d^3n\right)^n \Psi^0$$

$$\Psi^n(n) = \Psi^0 + \int G U\, d^3n_1 \left(\Psi^0 + \int G U \Psi^0\, d^3n_2 + \cdots + \left(\int G U\, d^3n\right)^{n-1} \Psi^0\right)$$

$$= \Psi^0 + \int G U \Psi^{n-1}\, d^3n_1$$

$$\boxed{\Psi^n(n) = \Psi^0 + \int G U \Psi^{n-1}\, d^3n_1}$$

b/
$$f(\theta) = -\frac{2m}{K\hbar^2} \int_0^\infty n\, V(n)\, \sin Kn\, dn$$

$$f(\theta) = +\frac{2m}{K\hbar^2} \int_0^a n\, V_0\, \sin Kn\, dn = -\frac{2mV_0}{K\hbar^2} \frac{d}{dK} \int_0^a \cos Kn\, dn$$

$$f(\theta) = -\frac{2mV_0}{K\hbar^2} \frac{d}{dK}\left(\frac{\sin Ka}{K}\right) = -\frac{2mV_0}{K\hbar^2}\left(-\frac{\sin Ka}{K^2} + \frac{a\cos Ka}{K}\right)$$

$$f(\theta) = \frac{2mV_0}{K^3\hbar^2}\left(Ka\cos Ka - \sin Ka\right)$$

$$= \frac{2mV_0}{K^3\hbar^2}\left[Ka\left(\cancel{1} - \frac{K^2a^2}{2}\right) - \left(\cancel{Ka} - \frac{1}{6}(Ka)^3\right)\right] = -\frac{2mV_0}{K^3\hbar^2}\frac{1}{3}(Ka)^3$$