Introduction

In this document, we will use the language of second quantization to diagonalize the Hamiltonian of the **"Ising Model"**. The main steps to follow are the following:

Second Quantization General Treatment for Spin Operator:

If we have a Hamiltonian of this form:

$$H = -\sum_i \left[S_i^x + \bar{\lambda} S_i^z S_{i+1}^z\right]; \qquad \text{The Ising model}$$

Step 1: Make the proper canonical transformation for spin operators, e.g.:

$$S^x \to S^z$$

Step 2: Write spin operators in terms of raising and lowering operators:

$$S^x = \frac{1}{2}\left(S^+ + S^-\right)$$

$$S^y = \frac{1}{2i}\left(S^+ - S^-\right)$$

$$S^z = \left(2S^+S^- - 1\right)$$

Step 3: Employ Jordan-Wigner transformation to transform into spinless fermion creation and annihilation operators:

$$c_i = \prod_{j=1}^{i-1}\left[-S_j^z\right] S_i^-$$

$$c_i^\dagger = S_i^+ \prod_{j=1}^{i-1}\left[-S_j^z\right]$$

Step 4: Transform into momentum space using Fourier transform:

$$c_j = \frac{1}{\sqrt{N}}\sum_q c_q e^{-iqj}$$

$$c_j^\dagger = \frac{1}{\sqrt{N}}\sum_q c_q^\dagger e^{iqj}$$

Step 5: Diagonalize the Hamiltonian via a unitary matrix and write transformed operators in terms of fermionic ones:

$$H = \underbrace{\begin{bmatrix} \vec{c}_q^{\,\dagger} & \vec{c}_{-q} \end{bmatrix} U^\dagger}_{\begin{bmatrix} \vec{\eta}_q^{\,\dagger} & \vec{\eta}_{-q} \end{bmatrix}} \underbrace{U h U^\dagger}_{D} \underbrace{U \begin{bmatrix} \vec{c}_q \\ \vec{c}_{-q}^{\,\dagger} \end{bmatrix}}_{\begin{bmatrix} \vec{\eta}_q & \vec{\eta}_{-q}^{\,\dagger} \end{bmatrix}^T} = 2\sum_q w_q \eta_q^\dagger \eta_q + E_0$$

The Ising Model

The ising model have this Hamiltonian:

$$H = -\sum_i \left[S_i^x + \bar{\lambda} S_i^z S_{i+1}^z\right]$$

Now I will use this canonical transformation to obtain a better Hamiltonian for my computation:

$$S^x \to S^z, \quad S^z \to -S^x$$

$$H = -\sum_i S_i^x - \bar{\lambda} \sum_i S_i^x S_{i+1}^x$$

Then, I will write them in terms of raising and lowering operators:

$$S^x = \frac{1}{2}\left(S^+ + S^-\right), \quad S^z = \left(2S^+S^- - 1\right)$$

The Hamiltonian have this form now:

$$H = N - 2\sum_i S_i^+ S_i^- - \bar{\lambda}\sum_i \left[S_i^+ S_{i+1}^+ + S_i^+ S_{i+1}^- + S_{i+1}^+ S_i^- + S_i^- S_{i+1}^-\right]$$

Now we will introduce Jordan-Wigner Transformation we can rewrite them as:

$$S_i^+ = c_i^\dagger \prod_{j=1}^{i-1} \left[-S_j^Z\right]$$

$$S_i^- = \prod_{j=1}^{i-1} \left[-S_j^Z\right] c_i$$

After using these transformation carefully, knowing $[S^z]^2 = \mathbb{1}$, and using the proper fermionic commutation relations, I found the following:

$$S_i^+ S_i^- = c_i^\dagger c_i$$

$$S_i^+ S_{i+1}^- = c_i^\dagger c_{i+1}$$

$$S_i^+ S_{i+1}^+ = c_i^\dagger c_{i+1}^\dagger$$

$$S_i^- S_{i+1}^+ = -c_i^\dagger c_{i+1}$$

$$S_i^- S_{i+1}^- = -c_i c_{i+1}$$

$$H = N - 2\sum_i c_i^\dagger c_i - \bar{\lambda}\sum_i \left[c_i^\dagger c_{i+1}^\dagger + c_i^\dagger c_{i+1} - c_i c_{i+1}^\dagger - c_i c_{i+1}\right]$$

Now we reached the fourth step where we need to use Fourier transform to move into momentum space. We need only to transform the first three, the rest can be obtained through hermitian conjugation:

$$knowing \sum_j e^{ij(q-q')} = N\delta_{qq'}$$

$$\sum_j c_j^\dagger c_j = \sum_q c_q^\dagger c_q$$

$$\sum_j c_j^\dagger c_{j+1} - c_j c_{j+1}^\dagger = \sum_j c_j^\dagger c_{j+1} + h.c. = \sum_q c_q^\dagger c_q \cos q$$

$$\sum_j c_j^\dagger c_{j+1}^\dagger - c_j c_{j+1} = \sum_j c_j^\dagger c_{j+1}^\dagger + h.c. = \sum_q e^{-iq} c_q^\dagger c_{-q}^\dagger - e^{iq} c_q c_{-q}$$

The Hamiltonian now becomes:

$$H = N - 2\sum_q (1 + \bar{\lambda}\cos q) c_q^\dagger c_q - \bar{\lambda}\sum_q (e^{-iq} c_q^\dagger c_{-q}^\dagger - e^{iq} c_q c_{-q})$$

We can make the sum over positive $q > 0$ only, to find the eigenvalues easier:

$$H = -2\sum_{q>0} (1 + \bar{\lambda}\cos q)(c_q^\dagger c_q - c_{-q}^\dagger c_{-q}) + 2i\bar{\lambda}\sum_{q>0} \sin q (c_q^\dagger c_{-q}^\dagger - c_q c_{-q})$$

Now we would like to write this Hamiltonian in matrix form:

$$H = -2 \begin{bmatrix} c_q^\dagger & c_{-q} \end{bmatrix} \begin{bmatrix} 1 + \bar{\lambda}\cos q & -i\bar{\lambda}\sin q \\ i\bar{\lambda}\sin q & -1 - \bar{\lambda}\cos q \end{bmatrix} \begin{bmatrix} c_q \\ c_{-q}^\dagger \end{bmatrix}$$

The eigenvalues of the matrix yield:

$$\begin{vmatrix} (1 + \bar{\lambda}\cos q) - w_q & -i\bar{\lambda}\sin q \\ i\bar{\lambda}\sin q & (-1 - \bar{\lambda}\cos q) - w_q \end{vmatrix} = 0 \implies w_q = \pm\sqrt{1 + 2\bar{\lambda}\cos q + \bar{\lambda}^2}$$

One can see that this type of Hamiltonians will yield this form of eigenvalues:

$$H = A \begin{bmatrix} c_q^\dagger & c_{-q} \end{bmatrix} \begin{bmatrix} \alpha & -i\beta \\ i\beta & -\alpha \end{bmatrix} \begin{bmatrix} c_q \\ c_{-q}^\dagger \end{bmatrix}$$

$$\left| H - w_q \mathbb{I} \right| = \begin{vmatrix} \alpha - w_q & -i\beta \\ i\beta & -\alpha - w_q \end{vmatrix} = 0 \implies w_q = \pm\sqrt{\alpha^2 + \beta^2}$$

Now we need to find the eigenvectors which will constitute the unitary matrix U:

$$U = \begin{bmatrix} | & | \\ V_1 & V_2 \\ | & | \end{bmatrix} = \begin{bmatrix} u_q & iv_q \\ iv_q & u_q \end{bmatrix}; \quad u_q = \frac{\alpha + w_q}{\sqrt{(\alpha + w_q)^2 + \beta^2}}; \quad v_q = \frac{\beta}{\sqrt{(\alpha + w_q)^2 + \beta^2}}$$

Where V_1 & V_2 are the first and second eigenvectors of the matrix. The Hamiltonian will have the following form:

$$H = \underbrace{\begin{bmatrix} c_q^\dagger & c_{-q} \end{bmatrix} U^\dagger}_{\begin{bmatrix} \eta_q^\dagger & \eta_{-q} \end{bmatrix}} \underbrace{UhU^\dagger}_{D} \underbrace{U \begin{bmatrix} c_q \\ c_{-q}^\dagger \end{bmatrix}}_{\begin{bmatrix} \eta_q & \eta_{-q}^\dagger \end{bmatrix}^T} = 2\sum_q w_q \eta_q^\dagger \eta_q + E_0$$