Why Dirac Notation? The Language of Quantum Computing

Think of it as Quantum Programming Syntax

- **Ket $|\psi\rangle$:** Column vector (quantum state) - like a variable holding quantum data
- **Bra $\langle\psi|$:** Row vector (conjugate transpose) - for computing probabilities
- **Bracket $\langle\phi|\psi\rangle$:** Inner product - returns complex number (amplitude)
- **Why use it?** Cleaner than matrices: $\langle 0|H|0\rangle$ vs $\begin{pmatrix}1 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}$

Notation Basics

Core Concepts

State Representation:

$$\text{Ket:}\quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\text{Bra:}\quad \langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1| = \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}$$

$$\text{Normalization:}\quad |\alpha|^2 + |\beta|^2 = 1$$

Computational Basis:

$$|0\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}, \quad |1\rangle = \begin{pmatrix}0\\ 1\end{pmatrix}$$

Key Operations:

- Gate application: $U|\psi\rangle$ applies unitary U to state $|\psi\rangle$
- Sequential gates: $U_2U_1|\psi\rangle$ applies U_1 first, then U_2
- Measurement probability: $P(i) = |\langle i|\psi\rangle|^2$
- Overlap: $\langle\phi|\psi\rangle$ gives amplitude
- Normalization: $\langle\psi|\psi\rangle = 1$

Tensor Products - Building Multi-Qubit Systems

Combining Qubits

Notation: $|\psi\rangle \otimes |\phi\rangle = |\psi\phi\rangle = |\psi\rangle|\phi\rangle$

2-Qubit Basis States:

Dirac	Vector
$|00\rangle = |0\rangle \otimes |0\rangle$	$(1,0,0,0)^T$
$|01\rangle = |0\rangle \otimes |1\rangle$	$(0,1,0,0)^T$
$|10\rangle = |1\rangle \otimes |0\rangle$	$(0,0,1,0)^T$
$|11\rangle = |1\rangle \otimes |1\rangle$	$(0,0,0,1)^T$

General 2-qubit state:

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

Tensor Product Rules:

- $(a|\psi\rangle) \otimes |\phi\rangle = a(|\psi\rangle \otimes |\phi\rangle)$
- $(|\psi_1\rangle + |\psi_2\rangle) \otimes |\phi\rangle = |\psi_1\rangle \otimes |\phi\rangle + |\psi_2\rangle \otimes |\phi\rangle$
- $A \otimes B$ creates block matrix

Common Single-Qubit States

The Building Blocks

Name	Dirac	Vector	Properties
Zero	$\|0\rangle$	$\begin{pmatrix}1\\0\end{pmatrix}$	Computational basis
One	$\|1\rangle$	$\begin{pmatrix}0\\1\end{pmatrix}$	Computational basis
Plus	$\|+\rangle = \frac{\|0\rangle+\|1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$H\|0\rangle = \|+\rangle$
Minus	$\|-\rangle = \frac{\|0\rangle-\|1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$	$H\|1\rangle = \|-\rangle$
Circular+	$\|i\rangle = \frac{\|0\rangle+i\|1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$	Y-basis eigenstate
Circular-	$\|-i\rangle = \frac{\|0\rangle-i\|1\rangle}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$	Y-basis eigenstate

These form 3 orthogonal pairs: $\{|0\rangle, |1\rangle\}$ (Z-basis), $\{|+\rangle, |-\rangle\}$ (X-basis), $\{|i\rangle, |-i\rangle\}$ (Y-basis)

Single-Qubit Gates

Fundamental Operations

Gate	Symbol	Matrix	Action
Pauli-X	X, σ_x	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$X\|0\rangle = \|1\rangle$, $X\|1\rangle = \|0\rangle$
Pauli-Y	Y, σ_y	$\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}$	$Y\|0\rangle = i\|1\rangle$, $Y\|1\rangle = -i\|0\rangle$
Pauli-Z	Z, σ_z	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$Z\|0\rangle = \|0\rangle$, $Z\|1\rangle = -\|1\rangle$
Hadamard	H	$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$	$H\|0\rangle = \|+\rangle$, $H\|1\rangle = \|-\rangle$
Phase	S	$\begin{pmatrix}1 & 0\\0 & i\end{pmatrix}$	$S\|0\rangle = \|0\rangle$, $S\|1\rangle = i\|1\rangle$
T gate	T	$\begin{pmatrix}1 & 0\\0 & e^{i\pi/4}\end{pmatrix}$	$T\|1\rangle = e^{i\pi/4}\|1\rangle$

Key insight: Gates are unitary matrices that preserve normalization: $U^\dagger U = I$

Two-Qubit Gates

Entangling Operations

Gate	Action	Matrix Size
CNOT	$\|00\rangle \to \|00\rangle$, $\|01\rangle \to \|01\rangle$, $\|10\rangle \to \|11\rangle$, $\|11\rangle \to \|10\rangle$	4×4
CZ	$\|11\rangle \to -\|11\rangle$, others unchanged	4×4
SWAP	$\|01\rangle \leftrightarrow \|10\rangle$, $\|00\rangle, \|11\rangle$ unchanged	4×4

CNOT Matrix: $\begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix}$ **Key:** CNOT creates entanglement from product states!

Entangled States

The Magic of Quantum Computing

Bell States (Maximally Entangled):

Name	**State**
$\|\Phi^+\rangle$	$\frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$
$\|\Phi^-\rangle$	$\frac{\|00\rangle-\|11\rangle}{\sqrt{2}}$
$\|\Psi^+\rangle$	$\frac{\|01\rangle+\|10\rangle}{\sqrt{2}}$
$\|\Psi^-\rangle$	$\frac{\|01\rangle-\|10\rangle}{\sqrt{2}}$

Key properties:

- Cannot write as $|\psi\rangle \otimes |\phi\rangle$ (not separable)
- Measuring one qubit instantly determines the other
- All Bell states are orthonormal
- Created by: $\text{CNOT}(H \otimes I)|00\rangle = |\Phi^+\rangle$

Inner Products - Computing Amplitudes

The Bracket Operation

Definition: $\langle\phi|\psi\rangle = \sum_i \phi_i^* \psi_i$
Key Properties:

- Returns complex number (amplitude)
- $\langle\psi|\psi\rangle = 1$ for normalized states
- $\langle 0|1\rangle = 0$ (orthogonal basis states)
- $\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$ (conjugate symmetry)
- $\langle\psi|U|\phi\rangle$ represents matrix element of U

Examples:

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$:
  $\langle\psi|\psi\rangle = \frac{1}{2}(1+1) = 1\checkmark$
- $\langle +|0\rangle = \frac{1}{\sqrt{2}}$
- $\langle +|1\rangle = \frac{1}{\sqrt{2}}$
- $\langle +|-\rangle = 0$ (orthogonal)

Outer Products and Projectors

Building Matrices from States

Outer Product: $|\psi\rangle\langle\phi|$ creates a matrix
Examples:

- $|0\rangle\langle 1| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$
- $|1\rangle\langle 0| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}$

Projectors: $P_i = |i\rangle\langle i|$

- $P_0 = |0\rangle\langle 0| = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$
- $P_1 = |1\rangle\langle 1| = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$
- Completeness: $P_0 + P_1 = I$
- Idempotent: $P_i^2 = P_i$

Applications:

- Density matrix:
  $\rho = |\psi\rangle\langle\psi|$ for pure state
- Measurement operators:
  $M_i = |i\rangle\langle i|$
- Gate construction:
  $X = |0\rangle\langle 1| + |1\rangle\langle 0|$
  $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$
  $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$

Measurements - How We Extract Information

The Fundamental Quantum Operation

Born Rule (Core of QC):

$$P(i) = |\langle i|\psi\rangle|^2$$

Example: For $|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$

- $P(0) = |\langle 0|\psi\rangle|^2 = \left|\frac{3}{5}\right|^2 = \frac{9}{25} = 36\%$
- $P(1) = |\langle 1|\psi\rangle|^2 = \left|\frac{4}{5}\right|^2 = \frac{16}{25} = 64\%$
- Verification: $P(0) + P(1) = 1\checkmark$

Measurement Process:

1. Calculate $P(i) = |\langle i|\psi\rangle|^2$
2. Outcome i with probability $P(i)$
3. State collapses: $|\psi\rangle \to |i\rangle$
4. Subsequent measurements give same result

General Observable:
For $A = \sum_i a_i |i\rangle\langle i|$:

- $\langle A\rangle = \langle\psi|A|\psi\rangle$
- $\text{Var}(A) = \langle A^2\rangle - \langle A\rangle^2$

Quick Reference Tables

Pauli Identities

- $X^2 = Y^2 = Z^2 = I$
- $XY = iZ$, $YZ = iX$, $ZX = iY$
- $XYZ = iI$
- $\{X, Y\} = 0$ (anticommute)
- $[X, Y] = 2iZ$ (commutator)
- $HXH = Z$, $HZH = X$

Useful Formulas

- $H = \frac{X+Z}{\sqrt{2}}$
- $|+\rangle = H|0\rangle$, $|-\rangle = H|1\rangle$
- $S = \sqrt{Z}$, $T = \sqrt{S}$
- $\text{CNOT} = (I \otimes H)CZ(I \otimes H)$
- Bell: $|\Phi^+\rangle = \text{CNOT}(H \otimes I)|00\rangle$

State Overlap Quick Check

$\langle\cdot\|\cdot\rangle$	$\|0\rangle$	$\|1\rangle$	$\|+\rangle$	$\|-\rangle$	$\|i\rangle$	$\|-i\rangle$
$\langle 0\|$	1	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
$\langle 1\|$	0	1	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{-i}{\sqrt{2}}$	$\frac{i}{\sqrt{2}}$
$\langle +\|$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	0	$\frac{1-i}{2}$	$\frac{1+i}{2}$