COE 530
Quantum Computers and Architecture

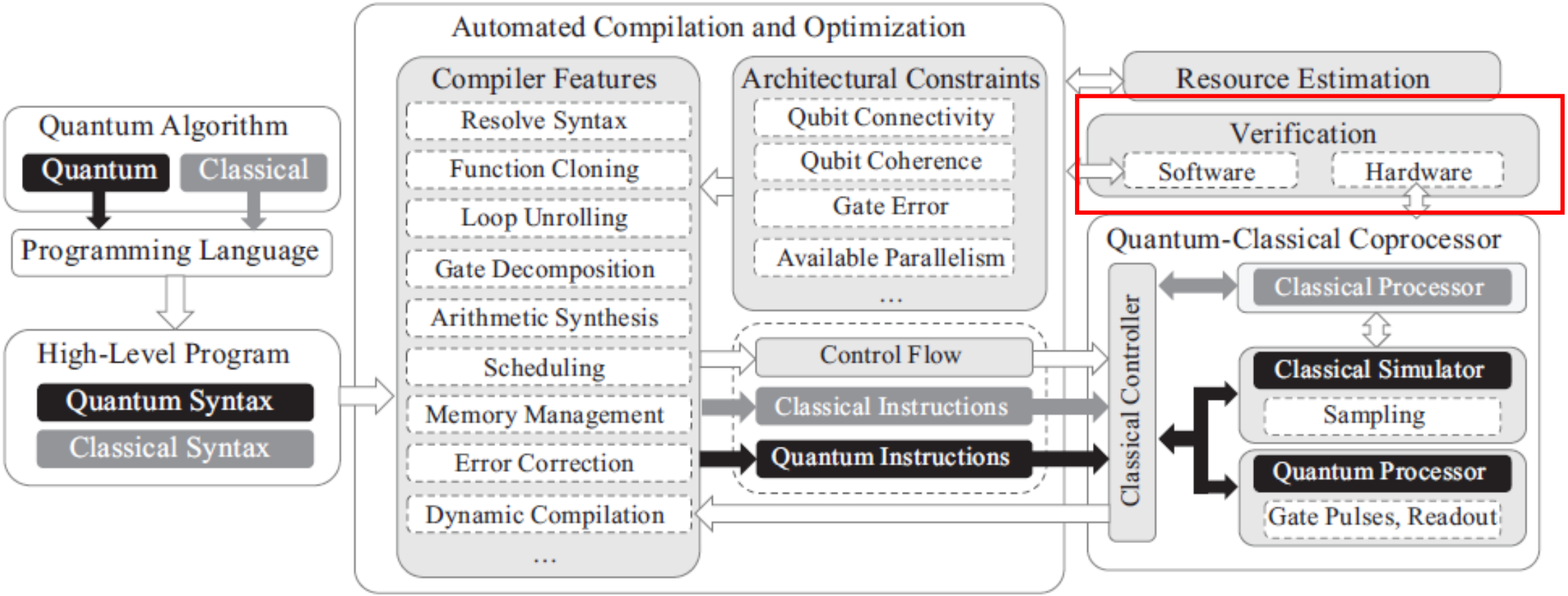
Lecture 9

Quantum Computing Systems – Quantum State Tomography

References:

Ch 5 and 8 Quantum Computer Systems

Quantum Compilation Flow



Quantum Program Debugging and Verification

- How do we know a quantum program implements the transformation as intended?
- Can we prevent programmers from writing code that violates some quantum properties?
- There are two possible notions of verification in quantum computation: *hardware verification* and *software verification*
  - **Software verification:** verify that a quantum program is bug-free and implements the desired transformation
  - **Hardware verification:** verifying that hardware can perform quantum logic operations as intended by a program

Software Verification

- The purpose of software verification is typically two-fold: verifying that
  1. *high-level programs* are bug-free,
  2. *compiler transformations* preserve logical equivalence

- How can achieve both hardware and software verification?
  - Classical simulators
  - What are the challenges of classical simulators?
  - There are also other techniques, including assertion-based and formal-methods

Recall - Classical Simulation of Quantum Computing

- Quantum computing simulations can be carried out through several classical simulation techniques and specialized quantum-inspired approaches:
  - State Vector Simulation
  - Unitary Simulation
  - Density Matrix
  - Stabilizer Simulations
  - Matrix Product States

- Each simulation method determines the internal representation of the quantum circuit and quantum operations

Strong vs. Weak Classical Simulation

- Strong Simulation
  - **Definition:** A classical algorithm that, given a quantum circuit C on n qubits, can compute the probability $p(x) = \left|\langle x|C|0^{\otimes n}\rangle\right|^2$ for every output string $x \in \{0,1\}^n$
  - Outputs the full probability distribution over all 2^n outcomes
  - Requires computing individual amplitudes $\langle x|C|0^{\otimes n}\rangle$
  - Exponential cost for general circuits

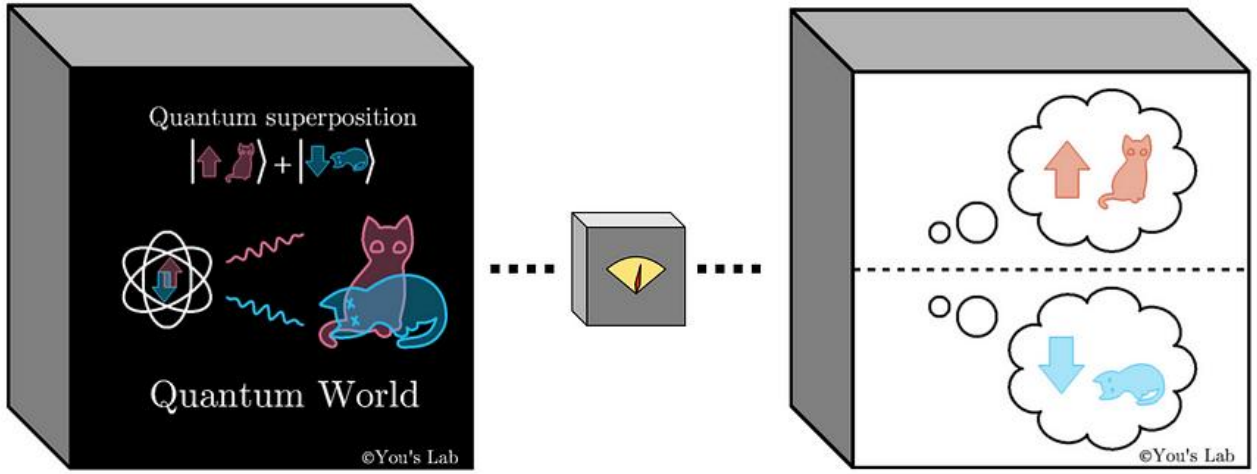
- Weak Simulation
  - **Definition:** A classical algorithm that, given a quantum circuit C on n qubits, can sample from a distribution D such that $|D - D_C| \leq \varepsilon$
  - Samples from a distribution ε-close to the true output
  - Does not require computing amplitudes
  - Can be poly-time for restricted circuits (e.g., Clifford via Gottesman-Knill)

Hardware Verification

- At a basic level, the behavior of quantum devices can be characterized through

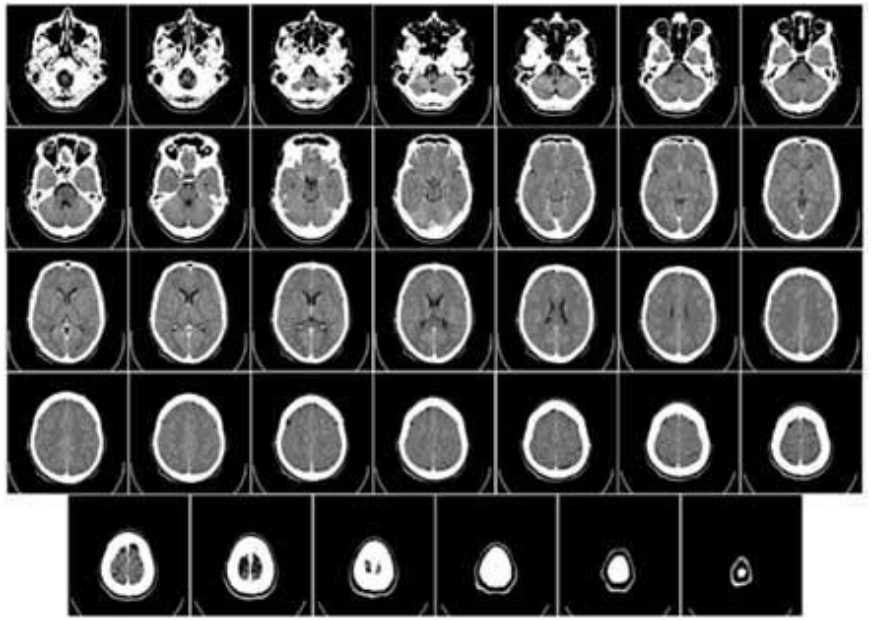
a process termed **quantum tomography**

- **Quantum tomography** refers to a set of techniques used to reconstruct unknown quantum objects from measurement data
- A quantum object can't be observed directly



Quantum State Tomography

- QST Reconstructs an unknown quantum state (density matrix) from measurement data
- Let ρ be an unknown quantum states of a system
  - Identical systems $X_1, X_2, \ldots, X_N$ are each independently prepared in the state ρ
  - Goal: approximate ρ by measuring $X_1, X_2, \ldots, X_N$
- QST is similar in concept to CT scans



https://www.radiologyinfo.org/en/info/safety-hiw_04

Quantum State Tomography (1/5)

Step 1: Prepare the Quantum System

- Prepare multiple identical copies of the quantum state ρ that you want to reconstruct.
- This is necessary because measuring a quantum state destroys it, so many copies are needed to collect sufficient statistics.
- **Example:** If you have a qubit in an unknown state, prepare thousands of identical qubits in the same state.

Quantum State Tomography (2/5)

Step 2: Choose Measurement Bases

- A single measurement basis is insufficient to reconstruct ρ due to the no-cloning theorem.
- For an n-qubit system, you need measurements in multiple bases.
- For a **single-qubit** system, common bases include:
  - **Computational Basis** (Z-basis): $\{|0\rangle, |1\rangle\}$
  - **X-basis**: $\{|+\rangle, |-\rangle\}$
  - **Y-basis**: $\{|i\rangle, |-i\rangle\}$
- For an **N-qubit system**, the number of required measurements grows exponentially, requiring measurements in tensor product bases (e.g., combinations of X, Y, and Z bases for each qubit).

Quantum State Tomography (3/5)

Step 3: Perform Measurements

- Measure each copy of the quantum state in one of the chosen bases
- Record the frequency of each measurement outcome

- **Example for a single qubit:** If measuring in the Z-basis:
  - p_0 = Probability of obtaining $|0\rangle$
  - p_1 = Probability of obtaining $|1\rangle$
  - Repeat for the X- and Y-bases

Quantum State Tomography (4/5)

Step 4: Estimate Expectation Values

- For a **single qubit**, the density matrix can be expressed in terms of Pauli matrices:

$$\rho = \frac{1}{2}(I + x\,\sigma_x + y\sigma_y + z\sigma_z)$$

where:

$$x = \langle\sigma_x\rangle, y = \langle\sigma_y\rangle\,, \mathrm{z} = \langle\sigma_z\rangle$$

The expectation values are computed as:

$$\langle\sigma_x\rangle = p_{+X} - p_{-X}, \langle\sigma_Y\rangle = p_{+Y} - p_{-Y}, \langle\sigma_Z\rangle = p_{+Z} - p_{-Z}$$

where p_{+B} and p_{-B} are the probabilities of measuring +1 and −1 in basis B.

Quantum State Tomography (5/5)

Step 5: Reconstruct the Density Matrix

- Using the collected measurement statistics, reconstruct the density matrix.
- For a **single qubit**, after computing x, y, z the estimated density matrix is:

$$\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

For **multi-qubit systems**, this extends to a $2^N \times 2^N$ density matrix.

QST Numerical Example (1/3)

Unknown state: $|\psi\rangle = (\frac{\sqrt{3}}{2})|0\rangle + (\frac{1}{2})|1\rangle$ (prepare 10,000 copies per basis)

Step 3a — Measure in the Z-basis $\{|0\rangle, |1\rangle\}$

- $P(|0\rangle) = |\frac{\sqrt{3}}{2}|^2 = ¾ = 0.75$ $P(|1\rangle) = |\frac{1}{2}|^2 = ¼ = 0.25$

Step 3b — Measure in the X-basis $\{|+\rangle, |-\rangle\}$ apply H before measurement

- $P(|+\rangle) = |\frac{\sqrt{3}+1}{2\sqrt{2}}|^2 = \frac{2+\sqrt{3}}{4} \approx 0.933$ $P(|-\rangle) = |\frac{\sqrt{3}-1}{2\sqrt{2}}|^2 = \frac{2-\sqrt{3}}{4} \approx 0.067$

Step 3c — Measure in the Y-basis $\{|i\rangle, |-i\rangle\}$ apply $S^\dagger H$ before measurement

- $|\psi\rangle$ has real amplitudes → $P(|i\rangle) = P(|-i\rangle) = 0.50$
- Check?

QST Numerical Example (2/3)

Step 4 — Estimate Expectation Values

Use: $\langle \sigma_B \rangle = P_{+B} - P_{-B}$ for each basis B

- **Z-basis:** $\langle \sigma_z \rangle = P(|0\rangle) - P(|1\rangle) = 0.75 - 0.25 = \mathbf{0.50}$
- **X-basis:** $\langle \sigma_x \rangle = P(|+\rangle) - P(|-\rangle) = 0.933 - 0.067 = \mathbf{0.866} \approx \sqrt{\mathbf{3}}/\mathbf{2}$
- **Y-basis:** $\langle \sigma_y \rangle = P(|R\rangle) - P(|L\rangle) = 0.50 - 0.50 = \mathbf{0.00}$

Bloch vector: **r** = (x, y, z) = ($\langle \sigma_x \rangle$, $\langle \sigma_y \rangle$, $\langle \sigma_z \rangle$) = **(0.866, 0.00, 0.50)**

$|\boldsymbol{r}| = \sqrt{0.866^2 + 0^2 + 0.50^2} = \sqrt{0.75 + 0 + 0.25} = \sqrt{1} = \mathbf{1.00}$ → pure state (on surface of Bloch sphere)

QST Numerical Example (3/3)

Step 5 — Reconstruct the Density Matrix

Using $\rho = \frac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$ with $x = 0.866,\ y = 0,\ z = 0.50$

$$\rho = \frac{1}{2}\begin{bmatrix} 1+0.5 & 0.866-0 \\ 0.866+0 & 1-0.5 \end{bmatrix}$$

$$\rho = \frac{1}{2}\begin{bmatrix} 1.5 & 0.866 \\ 0.866 & 0.5 \end{bmatrix}$$

$$\rho = \frac{1}{2}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$

Verification:

Tr(ρ) = 3/4 + 1/4 = 1 ✓ $\rho^\dagger = \rho$ ✓

Tr(ρ^2) = $0.75^2 + 2(0.433^2) + 0.25^2 = 1.00$ → **pure state** ✓

Matches $|\psi\rangle\langle\psi|$ for $|\psi\rangle = (\sqrt{3}/2)|0\rangle + (½)|1\rangle$ ✓ **QST successful!**

Types of Quantum Tomography

1. Quantum State Tomography (QST)
2. Quantum Process Tomography (QPT)
   - Characterizes an unknown quantum operation (quantum gate or noise process)
   - Uses a set of known input states and measures the output states
3. Quantum Detector Tomography (QDT)
   - Characterizes quantum measurement devices
   - Determines the positive operator-valued measure (POVM) elements describing a detector
4. Gate Set Tomography (GST)
   - Simultaneously reconstructs quantum states, processes, and measurements
   - Useful for diagnosing systematic errors in quantum hardware