King Fahd University of Petroleum and Minerals

College of Computing and Mathematics

Information and Computer Science Department

ICS 560: Foundations of Quantum Computing

Semester 251

Problem set 5

Show all your necessary steps:

1. Compute the quantum oracles for the other three unary functions ($f(x) = 0, f(x) = 1, f(x) = \bar{x}$, where $\bar{x}$ is the NOT of x) and observe that their matrices reveal them each to be unitary.
2. Compute the quantum oracles for the classical OR and XOR gates.
3. Suppose an input qubit and answer qubit are in the state $|x\rangle|+\rangle$, where x is a bit. If we apply the quantum phase oracle U_f to this state $|x\rangle|+\rangle$, what do we get if
   a) $f(x) = 0$ b) $f(x) = 1$.
4. Consider the following input qubit in the superposition state and an answer qubit in the state $|-\rangle$. find the output of the quantum oracle U_f.

$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

5. In Deutsch's algorithm, assume the last Hadamard gate is neglected, that is

$$|0\rangle \xrightarrow{H} |+\rangle \xrightarrow{U_f} \frac{1}{\sqrt{2}}\left[(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle\right]$$

   If the system measured, what possible states will be getting, and with what probabilities.
6. Suppose that $f(00) = f(01) = 0$, f $f(10) = f(11) = 1$. Apply the Deutsch-Josza algorithm and show that at least one of the first two qubits ends up as a 1.

$$|\psi\rangle = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} (-1)^{f(x)+\langle x,z\rangle} \; |z\rangle$$

7. Consider a function with two inputs such that $f(x) = 1$ for all x. Explicitly show that the Deutsch-Jozsa algorithm works in this case by generating the vector $|00\rangle$ as the final output.
8. Design a quantum circuit for implementing the quantum teleportation protocol: Alice wants to teleport a qubit in an unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. Assume Alice and Bob shared two entangled qubits in the state $|\varphi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

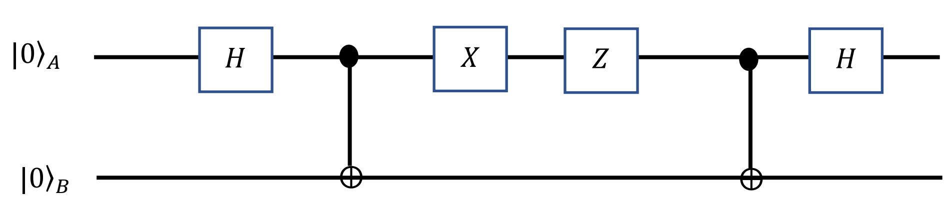
9. Alice wants to teleport a qubit in an unknown state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. Instead of sharing two entangled qubits in the $|\varphi^+\rangle$ state, they share two entangled qubits in the $|\psi^+\rangle$ state.
    a) If Alice applies $CNOT$ to her two qubits, followed by H to her left qubit, what is the state of the system becomes?
    b) Next, Alice measures both of her qubits. What values can she get, with what probabilities, and what does the state collapse to in each case?
    c) Finally, Alice tells Bob the results of her measurement. For each possible result, what should Bob do to his qubit so that is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, the state that Alice wanted to teleport to him?
10. Alice wants to teleport a qubit in an unknown state $|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$ to Bob. Instead of sharing two entangled qubits in a Bell state, they share three entangled qubits in the GHZ state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

    a) Find the initializing state of the teleportation protocol.
    b) Show that if Alice applies CNOT23 (recall the qubits are numbered left-to-right starting with 1), followed by CNOT12, followed by $H \otimes I \otimes I \otimes I$, the state of the system becomes:

$$\frac{1}{2}\big[|000\rangle(\alpha|0\rangle + \beta|1\rangle) + |010\rangle(\beta|0\rangle + \alpha|1\rangle) + |100\rangle(\alpha|0\rangle - \beta|1\rangle) + |110\rangle(-\beta|0\rangle + \alpha|1\rangle)\big].$$

    c) Next, Alice measures all three of her qubits. What values can she get, with what probabilities, and what does the state collapse to in each case?
    d) Finally, Alice tells Bob the results of her measurement. For each possible result, what should Bob do to his qubit so that is $\alpha|0\rangle + \beta|1\rangle$, the state that Alice wanted to teleport to him?
11. In the superdense coding, assume that Alice wants to send one of sixteen possible states to Bob.
    a) How many classical bits would Alice need to send to Bob?
    b) How many qubits would Alice need to send to Bob if they share entanglement?
    c) How many qubits total would it take, counting both Alice's and Bob's qubits?
12. Consider the following quantum circuit that implements superdense coding.



a) What result should Bob get for his measurement?

b) How would you modify the circuit so that Alice sends $|01\rangle$ to Bob? Justify your answer.

13. Given the following two density matrices of the two states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively. Determine which state reveal less information.

$$D_1 = \begin{bmatrix} \frac{9}{10} & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

14. Find the entropy of the state (Hint: use the fact $\lim_{x\to 0} x \log x = 0$)

$$|\varphi\rangle = \frac{2}{3}|0\rangle + \frac{\sqrt{5}}{3}|1\rangle$$

15. Consider the density operators for the state

$$|\varphi\rangle = \frac{\sqrt{3}}{\sqrt{7}}|0\rangle + \frac{2}{\sqrt{7}}|1\rangle$$

and the state

$$|\varphi\rangle = \frac{\sqrt{2}}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

Which state has a higher entropy?

16. Let $|\omega_1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - e^{i\pi}|1\rangle\right)$ and $|\omega_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi}|1\rangle\right)$. Assume that $|\omega_1\rangle$ is sent with probability $p_1 = \frac{3}{4}$ and $|\omega_2\rangle$ is sent with probability $p_2 = \frac{1}{4}$.
   a) Find the corresponding density matrix.
   b) Find the Von Neumann entropy of the mixed state.