Question 1

Given the complex numbers $Z_1 = 3 + 4i$ and $Z_2 = 1 - 2i$, find the following:

Solution:

(a) $Z_1 + Z_2 : 3 + 4i + 1 - 2i = 4 + 2i$

(b) $Z_1 - Z_2 : 3 + 4i - 1 + 2i = 2 + 6i$

(c) $Z_1 \times Z_2 : (3 + 4i) \times (1 - 2i) = 3 - 6i + 4i + 8 = 11 - 2i$

(d) $\frac{Z_2}{Z_1} = \frac{1-2i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{-5-10i}{25} = -\frac{1}{5} - \frac{2}{5}i$

Question 2

Find the modulus of $Z = -5 + 12i$.

Solution:

$$|-5 + 12i| = \sqrt{5^2 + 12^2} = 13$$

Question 3

Convert the complex number $Z = -5 - 5i$ from algebraic to polar form (ρ, θ).

Solution:

$$\begin{aligned}
\rho &= \sqrt{5^2 + 5^2} = 5\sqrt{2} \\
\theta &= \tan^{-1}(1) = \frac{\pi}{4} \implies \frac{5\pi}{4} \text{ (in 3rd quadrant)} \\
Z &= \left(5\sqrt{2}, \frac{5\pi}{4}\right)
\end{aligned}$$

Question 4

Write the complex number $Z = -4 + 4i\sqrt{3}$ in the Euler's form.

Solution:

$$\begin{aligned}
\rho &= \sqrt{4^2 + 3 \times 4^2} = 8 \\
\theta &= \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3} \implies \frac{2\pi}{3} \text{ (in 2nd quadrant)} \\
Z &= 8e^{i\frac{2\pi}{3}}
\end{aligned}$$

Question 5

Find the cube roots of the complex number $Z = 125e^{0i}$.

Solution:

$$\begin{aligned}
Z &= 125 + 0i = 125e^{0i} \\
z_0 &= 5 \\
z_1 &= 5e^{i\frac{2\pi}{3}} \\
z_2 &= 5e^{i\frac{4\pi}{3}}
\end{aligned}$$

Question 6

Given the complex number $Z = 3e^{i\frac{\pi}{4}}$, find Z^4.

Solution:

$$Z^4 = (3)^4 \, e^{i\frac{\pi}{4}\times 4} = 81e^{i\pi}$$

Question 7

Rotate the complex number $Z = 5e^{30^\circ i}$ by 90° counterclockwise. Express the resulting complex number in algebraic form.

Solution:

$$Z \times e^{90^\circ i} = 5e^{120^\circ i} = 5\left(\cos 120^\circ + i \sin 120^\circ\right) = -\frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

Question 8

Consider the vectors $v = \begin{bmatrix} 1+i \\ 2-i \\ 3 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ 1+i \\ 1-i \end{bmatrix}$, and $w = \begin{bmatrix} 3-i \\ i \\ 1 \end{bmatrix}$ in $\mathbb{C}^3$. Are these vectors linearly independent? Justify your answer.

Solution: By forming the matrix $[v|u|w]$:

$$\begin{bmatrix} 1+i & 2 & 3-i \\ 2-i & 1+i & i \\ 3 & 1-i & 1 \end{bmatrix}$$

The determinant is $-16 - 8i \neq 0$, therefore the vectors are linearly independent

Question 9

Given the vectors $v = \begin{bmatrix} 1+i \\ 2 \\ 3-i \end{bmatrix}$, $u = \begin{bmatrix} 2-i \\ i \\ 1 \end{bmatrix}$, calculate the inner product $\langle u, v \rangle$.

Solution:

$$\begin{aligned} \langle u, v \rangle &= \begin{bmatrix} 2+i & -i & 1 \end{bmatrix} \begin{bmatrix} 1+i \\ 2 \\ 3-i \end{bmatrix} \\ &= (2+i)(1+i) + (-i)(2) + (1)(3-i) \\ &= 2 + 2i + i - 1 - 2i + 3 - i \\ &= 4 \end{aligned}$$

Question 10

Find the norm (or length) of the vector $v = \begin{bmatrix} 2+i \\ -i \\ 1-i \end{bmatrix}$.

Solution:

$$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{2^2 + 1 + 1 + 1 + 1} = \sqrt{8} = 2\sqrt{2}$$

Question 11

Given the matrix $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$. Show that A is Hermitian matrix, Is this matrix A unitary? Justify your answer.

Solution:

$$\begin{aligned} A^\dagger = \overline{A}^T = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}^T &= \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = A \\ &\implies \text{A is Hermitian} \\ \det(A) = 1 \cdot 1 - i \cdot (-i) &= 1 - (-1) = 0 \\ &\implies \text{Singular} \implies \text{Not Unitary} \end{aligned}$$

Question 12

Show that the vectors $v = \begin{bmatrix} 1+i \\ 2-i \end{bmatrix}$ and $u = \begin{bmatrix} i \\ 1+2i \end{bmatrix}$ are linearly independent, then find the span of these vectors.

Solution:

$$\begin{aligned}
(1+i)c_1 + ic_2 = 0 &\implies c_2 = (i-1)c_1 \\
(2-i)c_1 + (1+2i)(i-1)c_1 = 0 & \\
(2-i)c_1 + (-1-2i)c_1 = 0 & \\
(-1-2i)c_1 = 0 &\implies c_1 = 0 \implies c_2 = 0
\end{aligned}$$

Alternatively:

$$\begin{aligned}
\det \begin{bmatrix} 1+i & i \\ 2-i & 1+2i \end{bmatrix} &= (1+i)(1+2i) - i(2-i) \\
&= -2 + i \neq 0 \implies \text{linearly independent} \\
\text{span}\{v, u\} &= \mathbb{C}^2
\end{aligned}$$

Question 13

Given the following vectors in $\mathbb{C}^2$

$v = \begin{bmatrix} \frac{3+i\sqrt{3}}{4} \\ \frac{1}{2} \end{bmatrix}$ and $u = \begin{bmatrix} \frac{1}{4} \\ x \end{bmatrix}$.

Find the value of x such that v and u are orthogonal.

Solution:

$$\begin{aligned}
\langle v, u \rangle = \begin{bmatrix} \frac{3-i\sqrt{3}}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ x \end{bmatrix} &= 0 \\
\frac{1}{16}(3 - i\sqrt{3}) + \frac{x}{2} &= 0 \\
x &= -\frac{1}{8}(3 - i\sqrt{3})
\end{aligned}$$

Question 14

Consider the matrix $A = \begin{bmatrix} 1+i & 2 \\ -i & 1-i \end{bmatrix}$, find the eigenvalues of A.

Solution:

$$\det(A - \lambda I) = 0$$
$$(1 + i - \lambda)(1 - i - \lambda) + 2i = 0$$
$$\lambda^2 - 2\lambda + (2 + 2i) = 0$$
$$\lambda = 1 \pm \sqrt{-1 - 2i}$$

Question 15

Find the eigenvalues and its associated eigenvectors of the following matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$.

Solution:

$$\det(A - \lambda I) = 0 \implies (2 - \lambda)^2 + 1 = 0$$
$$(2 - \lambda)^2 = -1 \implies 2 - \lambda = \pm i$$
$$\lambda = 2 \mp i$$

For eigenvalues:

$$\lambda = 2 - i : \quad \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies x_1 = ix_0 \implies v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$
$$\lambda = 2 + i : \quad \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\implies x_1 = -ix_0 \implies v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Question 16

Show that A is Hermitian if and only if $A^T = \overline{A}$.

Solution:

$$\text{Hermitian} \implies A = A^\dagger$$
$$\implies A = \overline{A}^T$$
$$\implies A^T = \left(\overline{A}^T\right)^T$$
$$\implies A^T = \overline{A} \quad \square$$

Question 17

Given the Hermitian matrix $A = \begin{bmatrix} 3 & 2-i \\ 2+i & 1 \end{bmatrix}$, find its eigenvalues. Verify that the eigenvalues are real.

Solution:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ (3-\lambda)(1-\lambda) - |2-i|^2 &= 0 \\ (3-\lambda)(1-\lambda) - 5 &= 0 \\ \lambda^2 - 4\lambda - 2 &= 0 \\ \lambda &= 2 \pm \sqrt{6} \end{aligned}$$

The eigenvalues are real, as expected for a Hermitian matrix.

Question 18

Let $B = \begin{bmatrix} 2 & 1+i \\ 1-i & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Compute the matrix $D = B \cdot C$ and determine whether D is Hermitian.

Solution:

$$\begin{aligned} D = B \cdot C &= \begin{bmatrix} 2 & 1+i \\ 1-i & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -(1+i) \\ 1-i & -2 \end{bmatrix} \\ D^\dagger &= \begin{bmatrix} 2 & 1+i \\ -(1-i) & -2 \end{bmatrix} \neq D \\ &\implies \text{Not Hermitian} \end{aligned}$$

Question 19

Compute the tensor product $A \otimes B$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1 \cdot B & 2 \cdot B \\ 3 \cdot B & 4 \cdot B \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 3 & 0 & 4 & 0 \end{bmatrix} \end{aligned}$$

Question 20

Compute the tensor product $A \otimes B$, where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Determine whether $A \otimes B$ is Hermitian.

Solution:

$$A \otimes B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 3 & 0 & 4 & 0 \end{bmatrix}$$

$$(A \otimes B)^\dagger = \overline{(A \otimes B)}^T = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 4 \\ 2 & 0 & 4 & 0 \end{bmatrix}$$

$$\neq A \otimes B \implies \text{Not Hermitian}$$