Question 1

How many bit strings of length 10 contain

(a) exactly four 1s?

(b) at most four 1s?

(c) at least four 1s?

(d) an equal number of 0s and 1s?

Solution:

(a) The number of bit strings with exactly four 1s is:

$$\binom{10}{4} = \frac{10!}{4!6!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

(b) At most four 1s means 0, 1, 2, 3, or 4 ones:

$$\begin{aligned}\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3} + \binom{10}{4} &= 1 + 10 + 45 + 120 + 210 \\ &= 386\end{aligned}$$

(c) At least four 1s is the complement of at most three 1s:

$$\begin{aligned}2^{10} - \left[\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}\right] &= 1024 - (1 + 10 + 45 + 120) \\ &= 1024 - 176 \\ &= 848\end{aligned}$$

(d) Equal number of 0s and 1s means exactly five 1s and five 0s:

$$\binom{10}{5} = \frac{10!}{5!5!} = 252$$

Question 2

What is the probability that a fair die never comes up an even number when it is rolled six times?

Solution:
For a fair die, the probability of rolling an odd number (1, 3, or 5) is $\frac{3}{6} = \frac{1}{2}$.
Since each roll is independent, the probability that all six rolls are odd is:

$$P(\text{never even}) = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

Question 3

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 1?

Solution:
Using the complement rule:

$$\begin{aligned} P(\text{at least one 1}) &= 1 - P(\text{all bits are 0}) \\ &= 1 - \left(\frac{1}{2}\right)^{10} \\ &= 1 - \frac{1}{1024} \\ &= \frac{1023}{1024} \end{aligned}$$

Question 4

What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?

Solution:
Let A = set of integers divisible by 5, and B = set of integers divisible by 7.

$$\begin{aligned} |A| &= \left\lfloor \frac{100}{5} \right\rfloor = 20 \\ |B| &= \left\lfloor \frac{100}{7} \right\rfloor = 14 \\ |A \cap B| &= \left\lfloor \frac{100}{35} \right\rfloor = 2 \text{ (divisible by lcm(5,7) = 35)} \end{aligned}$$

By the inclusion-exclusion principle:

$$|A \cup B| = |A| + |B| - |A \cap B| = 20 + 14 - 2 = 32$$

Therefore:

$$P = \frac{32}{100} = \frac{8}{25}$$

Question 5

A pair of dice is loaded. The probability that a 4 appears on the first die is 2/7, and the probability that a 3 appears on the second die is 2/7. Other outcomes for each die appear with probability 1/7. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

Solution:
The sum of 7 can occur with: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1).
For die 1: $P(4) = \frac{2}{7}$, all others have $P = \frac{1}{7}$

For die 2: $P(3) = \frac{2}{7}$, all others have $P = \frac{1}{7}$

$$\begin{aligned} P(\text{sum} = 7) &= P(1,6) + P(2,5) + P(3,4) + P(4,3) + P(5,2) + P(6,1) \\ &= \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \cdot \frac{2}{7} + \frac{2}{7} \cdot \frac{2}{7} + \frac{1}{7} \cdot \frac{1}{7} + \frac{1}{7} \cdot \frac{1}{7} \\ &= \frac{1+1+2+4+1+1}{49} \\ &= \frac{10}{49} \end{aligned}$$

Question 6

Let us assume that the particle is confined to $\{x_0, x_1, \cdots, x_5\}$ and the current state vector is $|\psi\rangle = [2 - i, 2i, 1 - i, 1, -2i, 2]^T$. What is the probability of finding the particle at position x_1?

Solution:
The probability of finding the particle at position x_1 is:

$$P(x_1) = \frac{|\langle x_1|\psi\rangle|^2}{\|\psi\|^2}$$

The amplitude at position x_1 is $2i$, so:

$$|\langle x_1|\psi\rangle|^2 = |2i|^2 = 4$$

The norm squared of $|\psi\rangle$ is:

$$\begin{aligned}\|\psi\|^2 &= |2 - i|^2 + |2i|^2 + |1 - i|^2 + |1|^2 + |-2i|^2 + |2|^2 \\ &= (4 + 1) + 4 + (1 + 1) + 1 + 4 + 4 \\ &= 5 + 4 + 2 + 1 + 4 + 4 \\ &= 20\end{aligned}$$

Therefore:

$$P(x_1) = \frac{4}{20} = \frac{1}{5}$$

Question 7

Do the vectors $|\psi\rangle = [1 + i, 2 - i]^T$ and $|\psi'\rangle = [2 + 2i, 1 - 2i]^T$ represent the same quantum state?

Solution:
Two vectors represent the same quantum state if and only if $|\psi'\rangle = c|\psi\rangle$ for some complex scalar c.
From the first component:

$$c = \frac{2 + 2i}{1 + i} = \frac{(2 + 2i)(1 - i)}{(1 + i)(1 - i)} = \frac{2 - 2i + 2i + 2}{2} = \frac{4}{2} = 2$$

Now check if this works for the second component:

$$c(2 - i) = 2(2 - i) = 4 - 2i \neq 1 - 2i$$

Since the scalar c does not work for both components, the vectors do **not** represent the same quantum state.

Question 8

Normalize the ket $|\psi\rangle = [3 - i, 2 + 6i, 7 - 8i, 13i, 0]^T$.

Solution:
First, compute the norm squared:

$$\begin{aligned}\|\psi\|^2 &= |3-i|^2 + |2+6i|^2 + |7-8i|^2 + |13i|^2 + |0|^2 \\ &= (9+1) + (4+36) + (49+64) + 169 + 0 \\ &= 10 + 40 + 113 + 169 + 0 \\ &= 332 \\ &= 4 \times 83\end{aligned}$$

Therefore:

$$\|\psi\| = \sqrt{332} = 2\sqrt{83}$$

The normalized ket is:

$$|\psi_{\text{norm}}\rangle = \frac{1}{2\sqrt{83}} \begin{bmatrix} 3-i \\ 2+6i \\ 7-8i \\ 13i \\ 0 \end{bmatrix} = \frac{1}{2\sqrt{83}} \begin{bmatrix} 3-i \\ 2+6i \\ 7-8i \\ 13i \\ 0 \end{bmatrix}$$

Question 9

Let the spinning electron's current state be $|\psi\rangle = 3i|\uparrow\rangle - 2|\downarrow\rangle$. Normalize the state and then find the probability that it will be detected in the up state.

Solution:
First, compute the norm squared:

$$\begin{aligned}\|\psi\|^2 &= |3i|^2 + |-2|^2 \\ &= 9 + 4 \\ &= 13\end{aligned}$$

Therefore $\|\psi\| = \sqrt{13}$.
The normalized state is:

$$|\psi_{\text{norm}}\rangle = \frac{1}{\sqrt{13}} (3i|\uparrow\rangle - 2|\downarrow\rangle)$$

The probability of detecting the electron in the up state is:

$$P(\uparrow) = |\langle\uparrow|\psi_{\text{norm}}\rangle|^2 = \left|\frac{3i}{\sqrt{13}}\right|^2 = \frac{|3i|^2}{13} = \frac{9}{13}$$

Question 10

Compute the amplitude of the transition from $|\psi\rangle = \frac{\sqrt{2}}{2}[i, -1]^T$ to $|\phi\rangle = \frac{\sqrt{2}}{2}[1, -i]^T$.

Solution:
The transition amplitude is given by the inner product $\langle\phi|\psi\rangle$.

First, find $\langle\phi|$:

$$\langle\phi| = \left(\frac{\sqrt{2}}{2}[1, -i]^T\right)^\dagger = \frac{\sqrt{2}}{2}[1, i]$$

Now compute the inner product:

$$\begin{aligned}
\langle\phi|\psi\rangle &= \frac{\sqrt{2}}{2}[1, i] \cdot \frac{\sqrt{2}}{2}\begin{bmatrix} i \\ -1 \end{bmatrix} \\
&= \frac{2}{4}\left(1 \cdot i + i \cdot (-1)\right) \\
&= \frac{1}{2}\left(i - i\right) \\
&= \frac{1}{2} \cdot 0 \\
&= 0
\end{aligned}$$

The transition amplitude is $\boxed{0}$, which means the states are orthogonal.

Question 11

Consider the following graph with 4 nodes (0, 1, 2, 3) where edges are labeled with $\frac{1}{\sqrt{2}}$.

(a) Construct the adjacency matrix of the above graph.

(b) Is the resulting matrix unitary?

Solution:
(a) From the graph structure (diamond shape):

- Node 0 connects to nodes 1 and 2
- Node 1 connects to nodes 0, 2, and 3
- Node 2 connects to nodes 0, 1, and 3
- Node 3 connects to nodes 1 and 2

The adjacency matrix is:

$$A = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

(b) To check if A is unitary, we need $A^\dagger A = I$.
Since A is real and symmetric, $A^\dagger = A$.
Computing A^2:

$$(A^2)_{00} = 0 + \frac{1}{2} + \frac{1}{2} + 0 = 1$$
$$(A^2)_{01} = 0 + 0 + \frac{1}{2} + 0 = \frac{1}{2}$$
$$(A^2)_{11} = \frac{1}{2} + 0 + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

Since $(A^2)_{11} = \frac{3}{2} \neq 1$, we have $A^2 \neq I$.
Therefore, A is **not unitary**.

Question 12

Consider the quantum system given by a graph with 8 nodes.

(a) Construct the adjacency matrix of the system.

(b) If the particle is at state $X = \left[\frac{1}{\sqrt{3}}, \sqrt{\frac{1}{5}}, 0, \frac{i}{\sqrt{15}}, 0, \frac{i}{\sqrt{15}}, \sqrt{\frac{2}{15}}, \sqrt{\frac{1}{5}}\right]^T$, find the state of X after one movement.

Solution:
(a) Analyzing the graph structure with 8 nodes (labeled 0-7):

From node 0: connects to nodes 1 and 4 with weight $\frac{1}{\sqrt{3}}$
From node 1: connects to node 2 with weight $\frac{1}{\sqrt{6}}$, to node 3 with weight $\frac{1}{\sqrt{6}}$, and self-loop with weight $\frac{1}{\sqrt{6}}$
From node 4: connects to node 5 with weight $\frac{1}{\sqrt{6}}$, to node 6 with weight $\frac{1}{\sqrt{6}}$, and self-loop with weight $\frac{1}{\sqrt{6}}$
Nodes 2, 3, 5, 6, 7 have self-loops with weight 1.
The adjacency matrix is:

$$A = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{6}} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) After one movement, the new state is $X' = AX$:

$$\begin{aligned} X' &= AX \\ &= \begin{bmatrix} \frac{1}{\sqrt{3}} \cdot \sqrt{\frac{1}{5}} + \frac{1}{\sqrt{3}} \cdot 0 \\ \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} \cdot \sqrt{\frac{1}{5}} + \frac{1}{\sqrt{6}} \cdot 0 + \frac{1}{\sqrt{6}} \cdot \frac{i}{\sqrt{15}} \\ \frac{1}{\sqrt{6}} \cdot \sqrt{\frac{1}{5}} + 1 \cdot 0 \\ \frac{1}{\sqrt{6}} \cdot \sqrt{\frac{1}{5}} + 1 \cdot \frac{i}{\sqrt{15}} \\ \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}} \cdot 0 + \frac{1}{\sqrt{6}} \cdot \frac{i}{\sqrt{15}} + \frac{1}{\sqrt{6}} \cdot \sqrt{\frac{2}{15}} \\ \frac{1}{\sqrt{6}} \cdot 0 + 1 \cdot \frac{i}{\sqrt{15}} \\ \frac{1}{\sqrt{6}} \cdot 0 + 1 \cdot \sqrt{\frac{2}{15}} \\ 1 \cdot \sqrt{\frac{1}{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{15}} \\ \frac{1}{3} + \frac{1}{\sqrt{30}} + \frac{i}{6\sqrt{5}} \\ \frac{1}{\sqrt{30}} \\ \frac{1}{\sqrt{30}} + \frac{i}{\sqrt{15}} \\ \frac{1}{3} + \frac{i}{6\sqrt{5}} + \sqrt{\frac{2}{90}} \\ \frac{i}{\sqrt{15}} \\ \sqrt{\frac{2}{15}} \\ \sqrt{\frac{1}{5}} \end{bmatrix} \end{aligned}$$