Question 1

Compute $\langle 1+0|1-0\rangle$.

Solution:
These are product states: $|1+0\rangle = |1\rangle \otimes |+\rangle \otimes |0\rangle$ and $|1-0\rangle = |1\rangle \otimes |-\rangle \otimes |0\rangle$.
For product states, the inner product factors:

$$\begin{aligned}\langle 1+0|1-0\rangle &= \langle 1|1\rangle \cdot \langle +|-\rangle \cdot \langle 0|0\rangle \\ &= 1 \cdot 0 \cdot 1 \\ &= 0\end{aligned}$$

Since $\langle +|-\rangle = \frac{1}{2}(\langle 0| + \langle 1|)(|0\rangle - |1\rangle) = \frac{1}{2}(1-1) = 0$.

Question 2

Consider the two-qubit state

$$|\psi\rangle = \frac{1}{4}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{4}|11\rangle$$

If you measure only the left qubit, what are the resulting states, and with what probabilities?

Solution:
Group terms by the left qubit value:

$$|\psi\rangle = |0\rangle \otimes \left(\frac{1}{4}|0\rangle + \frac{1}{2}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{4}|1\rangle\right)$$

Probability of measuring left qubit in state $|0\rangle$:

$$P(|0\rangle) = \left|\frac{1}{4}\right|^2 + \left|\frac{1}{2}\right|^2 = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$$

Post-measurement state:

$$|\psi_0\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle\right)$$

Probability of measuring left qubit in state $|1\rangle$:

$$P(|1\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{\sqrt{3}}{4}\right|^2 = \frac{1}{2} + \frac{3}{16} = \frac{11}{16}$$

Post-measurement state:

$$|\psi_1\rangle = |1\rangle \otimes \left(\frac{4}{\sqrt{22}}|0\rangle + \frac{\sqrt{3}}{\sqrt{11}}|1\rangle\right)$$

Question 3

Consider the three-qubit state

$$|\psi\rangle = \frac{1}{6}|000\rangle + \frac{1}{3\sqrt{2}}|001\rangle + \frac{1}{\sqrt{6}}|010\rangle + \frac{1}{2}|011\rangle + \frac{1}{6}|100\rangle + \frac{1}{3}|101\rangle + \frac{1}{6}|110\rangle + \frac{1}{\sqrt{3}}|111\rangle$$

If you measure only the left and right qubits, but not the middle qubit, what are the resulting states, and with what probabilities?

Solution:
Group by left and right qubit values:
Case $|0\rangle_L|0\rangle_R$: Terms: $\frac{1}{6}|000\rangle + \frac{1}{\sqrt{6}}|010\rangle$

$$P(|0\rangle_L|0\rangle_R) = \frac{1}{36} + \frac{1}{6} = \frac{7}{36}, \quad |\psi\rangle = |0\rangle\left(\frac{1}{\sqrt{7}}|0\rangle + \frac{\sqrt{6}}{\sqrt{7}}|1\rangle\right)|0\rangle$$

Case $|0\rangle_L|1\rangle_R$: Terms: $\frac{1}{3\sqrt{2}}|001\rangle + \frac{1}{2}|011\rangle$

$$P(|0\rangle_L|1\rangle_R) = \frac{1}{18} + \frac{1}{4} = \frac{11}{36}, \quad |\psi\rangle = |0\rangle\left(\frac{2}{\sqrt{11}}|0\rangle + \frac{3}{\sqrt{22}}|1\rangle\right)|1\rangle$$

Case $|1\rangle_L|0\rangle_R$: Terms: $\frac{1}{6}|100\rangle + \frac{1}{6}|110\rangle$

$$P(|1\rangle_L|0\rangle_R) = \frac{1}{36} + \frac{1}{36} = \frac{2}{36}, \quad |\psi\rangle = |1\rangle|+\rangle|0\rangle$$

Case $|1\rangle_L|1\rangle_R$: Terms: $\frac{1}{3}|101\rangle + \frac{1}{\sqrt{3}}|111\rangle$

$$P(|1\rangle_L|1\rangle_R) = \frac{1}{9} + \frac{1}{3} = \frac{16}{36}, \quad |\psi\rangle = |1\rangle\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right)|1\rangle$$

Question 4

Are each of the following states separable or entangled? If it is a product state, give the factorization.

(a) $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

(b) $\frac{1}{\sqrt{2}}(|10\rangle - i|11\rangle)$

Solution:
For $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, the state is separable iff $ad = bc$.
(a) $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$$(a, b, c, d) = \left(0,\ \frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ 0\right) \quad \Rightarrow \quad ad = 0,\ bc = -\frac{1}{2}$$

Since $ad \neq bc$: **Entangled** (Bell state $|\Psi^-\rangle$).
(b) $|\psi\rangle = \frac{1}{\sqrt{2}}(|10\rangle - i|11\rangle)$

$$(a, b, c, d) = \left(0,\ 0,\ \frac{1}{\sqrt{2}},\ -\frac{i}{\sqrt{2}}\right) \quad \Rightarrow \quad ad = 0,\ bc = 0$$

Since $ad = bc$: **Separable** $= |1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

Question 5

Are each of the following states separable or entangled? If it is a separable state, give the factorization.

(a) $\frac{1}{4}(3|00\rangle - \sqrt{3}|01\rangle + \sqrt{3}|10\rangle - |11\rangle)$

(b) $\frac{1}{\sqrt{3}}|0+\rangle + \sqrt{\frac{2}{3}}|1-\rangle$

Solution:
For $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$, the state is separable iff $ad = bc$.
(a) $|\psi\rangle = \frac{1}{4}(3|00\rangle - \sqrt{3}|01\rangle + \sqrt{3}|10\rangle - |11\rangle)$

$$(a, b, c, d) = \left(\frac{3}{4},\ -\frac{\sqrt{3}}{4},\ \frac{\sqrt{3}}{4},\ -\frac{1}{4}\right) \quad \Rightarrow \quad ad = bc = -\frac{3}{16}$$

Separable $= \left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle\right) \otimes \left(\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right)$
(b) $|\psi\rangle = \frac{1}{\sqrt{3}}|0+\rangle + \sqrt{\frac{2}{3}}|1-\rangle = \frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle$

$$(a, b, c, d) = \left(\frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{6}},\ \frac{1}{\sqrt{3}},\ -\frac{1}{\sqrt{3}}\right) \quad \Rightarrow \quad ad = -\frac{1}{\sqrt{18}},\ bc = \frac{1}{\sqrt{18}}$$

Since $ad \neq bc$: **Entangled**.

Question 6

Prove the following circuit identities by finding the matrix representation of each circuit.

(a) $\text{CNOT}(I \otimes X) = (I \otimes X)\text{CNOT}$

(b) $\text{CNOT}(Z \otimes I) = (Z \otimes I)\text{CNOT}$

Solution:

Recall: $\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(a) $I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$$\text{CNOT}(I \otimes X) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(I \otimes X)\text{CNOT} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

(b) $Z \otimes I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$$\text{CNOT}(Z \otimes I) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$(Z \otimes I)\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \checkmark$$

Question 7

The controlled-U (CU) gate applies some quantum gate U to the right qubit if the left qubit is 1. Write the controlled-Z and controlled-Y gates as a matrix.

Solution:
The controlled-U gate has the form:

$$CU = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{bmatrix}$$

Controlled-Z (CZ): With $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

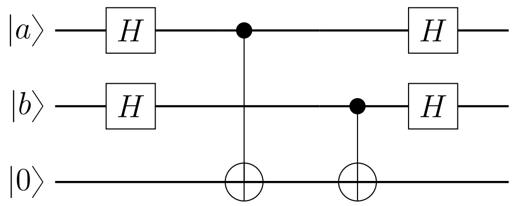
$$CZ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Controlled-Y (CY): With $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$$CY = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}$$

Question 9

Consider the following quantum circuit:



(a) If $|a\rangle = |-\rangle$ and $|b\rangle = |+\rangle$, find the resulting state at the end of the circuit.

(b) If $|a\rangle = |-\rangle$ and $|b\rangle = |-\rangle$, find the resulting state at the end of the circuit.

Solution:
Recall: $H|\pm\rangle = |0/1\rangle$ and $H|0/1\rangle = |\pm\rangle$.
(a) $|a\rangle = |-\rangle$, $|b\rangle = |+\rangle$

$$\begin{aligned}
|-\rangle|+\rangle|0\rangle &\xrightarrow{H\otimes H\otimes I} |1\rangle|0\rangle|0\rangle \\
&\xrightarrow{\text{CNOT}_1} |1\rangle|0\rangle|1\rangle \\
&\xrightarrow{\text{CNOT}_2} |1\rangle|0\rangle|1\rangle \\
&\xrightarrow{H\otimes H\otimes I} \boxed{|-\rangle|+\rangle|1\rangle}
\end{aligned}$$

(b) $|a\rangle = |-\rangle$, $|b\rangle = |-\rangle$

$$\begin{aligned}
|-\rangle|-\rangle|0\rangle &\xrightarrow{H\otimes H\otimes I} |1\rangle|1\rangle|0\rangle \\
&\xrightarrow{\text{CNOT}_1} |1\rangle|1\rangle|1\rangle \\
&\xrightarrow{\text{CNOT}_2} |1\rangle|1\rangle|0\rangle \\
&\xrightarrow{H\otimes H\otimes I} \boxed{|-\rangle|-\rangle|0\rangle}
\end{aligned}$$