# 1.4 Inner product and Hilbert Space

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**Definition 2.4.1** *An* **inner product** *(also called a* **dot product** *or* **scalar product***) on a complex vector space* $\mathbb{V}$ *is a function* $f: \mathbb{V}_1 \times \mathbb{V}_2 \longrightarrow \mathbb{C}$

$$f(v_1, v_2) = \langle v_1, v_2 \rangle = v_1^{\dagger} * v_2 = \sum_{i=1}^{n} \overline{c_i}\, c'_i$$

where

$$v_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad v_2 = \begin{bmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \overline{c_1} & \overline{c_2} & \cdots & \overline{c_n} \end{bmatrix}}_{1\times n} \underbrace{\begin{bmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{bmatrix}}_{n\times 1}$$

$$= \overline{c_1} c'_1 + \overline{c_2} \cdot c'_2 + \cdots + \overline{c_n} c'_n$$

Example 1: Find $\langle v_1, v_2 \rangle$

$$v_1 = \begin{bmatrix} 2+i \\ 3-2i \end{bmatrix} \qquad v_1 = \begin{bmatrix} -3 \\ 2i \end{bmatrix}$$

$$\langle v_1, v_2 \rangle = \begin{bmatrix} 2-i & 3+2i \end{bmatrix} \begin{bmatrix} -3 \\ 2i \end{bmatrix}$$

$$= -6 + 3i + 6i - 4 = -10 + 9i$$

The inner product function $f: \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{C}$ satisfies the following conditions :
For all $v, v_1, v_2, and\ v_3 \in \mathbb{V}$ and $c \in \mathbb{C}$

(i) *Nondegenerate:*

$$\langle V, V \rangle \geq 0,$$

$$\langle V, V \rangle = 0 \text{ if and only if } V = \mathbf{0}$$

*(i.e., the only time it "degenerates" is when it is 0).*

$$V = \begin{bmatrix} 2+i \\ 3-i \end{bmatrix} \Rightarrow \langle V, V \rangle = \begin{bmatrix} 2-i & 3+i \end{bmatrix} \begin{bmatrix} 2+i \\ 3-i \end{bmatrix}$$

$$= 5 + 10 = \underline{\underline{15}}$$

(ii) *Respects addition:*

$$\langle V_1 + V_2, V_3 \rangle = \langle V_1, V_3 \rangle + \langle V_2, V_3 \rangle,$$

$$\langle V_1, V_2 + V_3 \rangle = \langle V_1, V_2 \rangle + \langle V_1, V_3 \rangle.$$

$$\langle v_1 + v_2, v_3 \rangle = (v_1 + v_2)^{\dagger} * v_3$$

$$= (v_1^{\dagger} + v_2^{\dagger}) * v_3$$

$$= v_1^{\dagger} * v_3 + v_2^{\dagger} * v_3$$

$$= \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$$

(iii) *Respects scalar multiplication:*

$$\langle c \cdot V_1, V_2 \rangle = \overline{c} \times \langle V_1, V_2 \rangle,$$

$$\langle c \cdot v_1, v_2 \rangle = (c v_1)^{\dagger} \cdot v_2 = \overline{c}\,(v_1^{\dagger} \cdot v_2)$$

$$= \overline{c} \times \langle v_1, v_2 \rangle$$

$$\langle V_1, c \cdot V_2 \rangle = c \times \langle V_1, V_2 \rangle.$$

$$\langle V_1, c V_2 \rangle$$

$$= V_1^{\dagger} * c * V_2$$

$$= c \times (V_1^{\dagger} * V_2)$$

$$= c \times \langle V_1, V_2 \rangle$$

(iv) *Skew symmetric:*

$$\langle V_1, V_2 \rangle = \overline{\langle V_2, V_1 \rangle}.$$

$$\overline{(V_2^{\dagger} \cdot V_1)} = \overline{V_2^{\dagger}}\ \overline{V_1} = (V_2^{T} \times \overline{V_1})^{T} \qquad (\overline{V_2^{\dagger}}) = (V_2^{T})$$

$$(\overline{V_1})^{T} \cdot (V_2^{T})^{T}$$

$$= V_1^{\dagger}\ V_2 = \langle V_1, V_2 \rangle$$

Definition (**Trace**): the trace of a square matrix C is given by the sum of the diagonal elements

$$Trace(C) = \sum_{i=0}^{n} c[i, i]$$

Example 2: Find the trace of the matrix

$$A = \begin{bmatrix} 3-i & 2 & i \\ -i & 0 & 4 \\ 7 & 2-i & i \end{bmatrix} \Longrightarrow Trace(A) = 3 - i + 0 + i = \underline{\underline{3}}$$

Definition : The inner product given for matrices $A, B \in \underline{\mathbb{C}}^{m\times n}$

$$\langle A, B\rangle = Trace(A^{\dagger} * B)$$

Example 3: Find the inner product of two matrices

$$A = \begin{bmatrix} i & 2-i \\ 3 & -i \end{bmatrix} \qquad B = \begin{bmatrix} 1+i & 4 \\ 5-i & 2+3i \end{bmatrix}$$

$$A^{\dagger} * B = \begin{bmatrix} -i & 3 \\ 2+i & i \end{bmatrix} \begin{bmatrix} 1+i & 4 \\ 5-i & 2+3i \end{bmatrix} = \begin{array}{cc} 1\times 1 & 1\times 2 \\ \end{array} \begin{bmatrix} \boxed{16-4i} & \\ 2\times 1 & \boxed{5+6i}_{\,2\times 2} \end{bmatrix}$$

$$\langle A, B\rangle = 16 - 4i + 5 + 6i = 21 + 2i$$

■ $\mathbb{R}^{n\times n}$ has an inner product given for matrices $A, B \in \mathbb{R}^{n\times n}$ as

$$\langle A, B\rangle = Trace(A^T \star B),$$

**Definition 2.4.3** *For every complex inner product space* $\mathbb{V}, \langle -, -\rangle$, *we can define a* **norm** *or* **length** *which is a function*

$V \in \mathbb{V}$ $\quad |V|$

$$|\quad| : \mathbb{V} \longrightarrow \mathbb{R}$$

*defined as* $|V| = \sqrt{\langle V, V\rangle}$.

**Exercise 2.4.5** Calculate the norm of $V = [4+3i, 6-4i, 12-7i, 13i]^T$. $= \begin{bmatrix} 4+3i \\ 6-4i \\ 12-7i \\ 13i \end{bmatrix}$

$$|V| = \sqrt{\langle V, V\rangle} = \sqrt{4^2+3^2+6^2+4^2+12^2+7^2+13^2} = 20952$$

$$\begin{bmatrix} 4-3i & 6+4i & 12+7i & -13i \end{bmatrix} \begin{bmatrix} 4+3i \\ 6-4i \\ 12-7i \\ 13i \end{bmatrix}$$

From the properties of an inner product space, it follows that a norm has the following properties for all $V, W \in \mathbb{V}$ and $c \in \mathbb{C}$:

$\sqrt{\langle V, V\rangle} \geq 0$

(i) Norm is nondegenerate: $|V| > 0$ if $V \neq \mathbf{0}$ and $|\mathbf{0}| = 0$.
(ii) Norm satisfies the **triangle inequality**: $|V + W| \leq |V| + |W|$.
(iii) Norm respects scalar multiplication: $|c \cdot V| = |c| \times |V|$. $= \sqrt{c * \bar{c}}$ $\quad \sqrt{\langle V, V\rangle}$

(ii) Norm satisfies the **triangle inequality**: $|V + W| \leq |V| + |W|$.
(iii) Norm respects scalar multiplication: $|c \cdot V| = |c| \times |V|$. $= \sqrt{c \times \bar{c}}$ $\quad \sqrt{\langle V, V \rangle}$

Example 6: Let $A = \begin{bmatrix} 2i & -1 \\ 3+i & 5 \end{bmatrix}$, find the norm of $A$

$A^{+} * A = \begin{bmatrix} -2i & 3-i \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 2i & -1 \\ 3+i & 5 \end{bmatrix}$

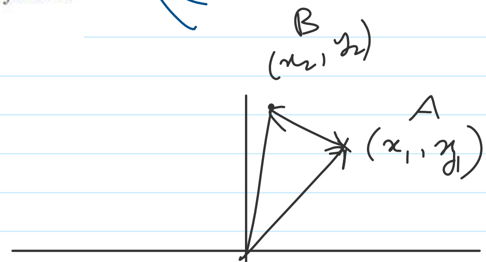
$|A| = \sqrt{\langle A, A \rangle} = \sqrt{\mathrm{Trace}(A^{+} * A)} = \sqrt{40} = 2\sqrt{10}$ $\quad = \begin{bmatrix} 14 & \\ & 26 \end{bmatrix}$

**Definition 2.4.4** *For every complex inner product space* $(\mathbb{V}, \langle\ ,\ \rangle)$, *we can define a* **distance function**

$$d(\ ,\ ) : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R},$$

*where*

$$d(V_1, V_2) = |V_1 - V_2| = \sqrt{\langle V_1 - V_2, V_1 - V_2 \rangle}.$$




$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$2i - 4 - i$

Example 2.4.5: Find the distance between $v_1 = \begin{bmatrix} 2i \\ 1-i \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 4+i \\ 7+i \\ 2-i \end{bmatrix}$

$1 - i - 7 - i$

$d(v_1, v_2) = |v_1 - v_2|$

$= \sqrt{\langle v_1 - v_2, v_1 - v_2 \rangle}$

$= \sqrt{4^2 + 1^2 + 6^2 + 2^2 + 1^2 + 1^2} = \sqrt{59}$

$v_1 - v_2 = \begin{bmatrix} -4 + i \\ -6 - 2i \\ 1 + i \end{bmatrix}$

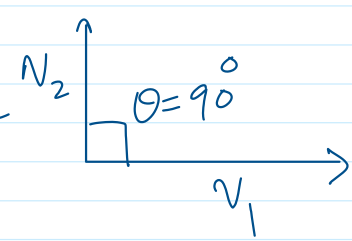
**Note**: The distance function satisfied the following properties, for $V, U, W \in \mathbb{V}$

(i) Distance is nondegenerate: $d(V, W) > 0$ if $V \neq W$ and $d(V, V) = 0$.
(ii) Distance satisfies the **triangle inequality**: $d(U, V) \leq d(U, W) + d(W, V)$.
(iii) Distance is symmetric: $d(V, W) = d(W, V)$.

In Real Number space $\langle V, V' \rangle = |V||V'| \cos\theta$, where $\theta$ is the angle between $V$ and $V'$.

**Definition 2.4.5** *Two vectors* $V_1$ *and* $V_2$ *in an inner product space* $\mathbb{V}$ *are* **orthogonal** *if* $\langle V_1, V_2 \rangle = 0$.

$\langle v_1, v_2 \rangle = |v_1||v_2| \underbrace{\cos(90^\circ)}_{=0} = 0$




Example 7: Determine if each pair of states is orthogonal or not.

a) $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\quad$ b) $v_1 = \begin{bmatrix} \frac{1-\sqrt{3}i}{4} \\ \frac{\sqrt{2}+i}{2} \end{bmatrix}$ and $v_2 = \begin{bmatrix} \frac{2+i}{2} \\ \frac{-1+\sqrt{3}i}{4} \end{bmatrix}$

ⓐ $\langle V_1, V_2 \rangle = [0 \quad 1]\begin{bmatrix}1\\-i\end{bmatrix} = -i \neq 0$ Not orthognal.

ⓑ

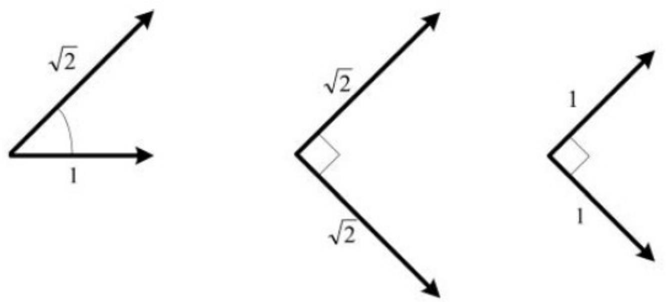
**Definition 2.4.6** *A basis $\mathcal{B} = \{V_0, V_1, \ldots, V_{n-1}\}$ for an inner product space $\mathbb{V}$ is called an* **orthogonal basis** *if the vectors are pairwise orthogonal to each other, i.e., $j \neq k$ implies $\langle V_j, V_k \rangle = 0$. An orthogonal basis is called an* **orthonormal basis** *if every vector in the basis is of norm 1, i.e.,*

$$\langle V_j, V_k \rangle = \delta_{j,k} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

$\Rightarrow \langle V_2, V_2 \rangle = 1$ $\quad |V_2| = \sqrt{1} = 1$

$\langle V_2, V_3 \rangle = 0$

*$\delta_{j,k}$ is called the* **Kronecker delta function**.




(i) Not orthogonal (ii) Orthogonal but not orthonormal (iii) Orthonormal

Example 8: Consider the three bases, determine the orthogonal and orthonormal

(i) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix},$

(ii) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix},$

(iii) $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}.$

**Exercise 2.4.8** Let $V = [3, -1, 0]^T$ and $V' = [2, -2, 1]^T$. Calculate the angle $\theta$ between these two vectors. ■

**Proposition 2.4.1** In $\mathbb{C}^n$, we also have that any $V$ can be written as

$$V = \overset{c_0}{\langle E_0, V\rangle} E_0 + \overset{c_1}{\langle E_1, V\rangle} E_1 + \cdots + \overset{c_n}{\langle E_{n-1}, V\rangle} E_{n-1}.$$

It must be stressed that this is true for any orthonormal basis, not just the canonical one.

**Definition 2.4.9** *A* **Hilbert space** *is a complex inner product space that is complete.*

**Proposition 2.4.2** Every inner product on a *finite*-dimensional complex vector space is automatically complete; hence, every finite-dimensional complex vector space with an inner product is automatically a Hilbert space.