Revision of Complex Numbers

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"*Associated to any isolated physical system is a complex vector space with inner product (i.e. a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.*" (Nielsen and Chuang).

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\alpha, \beta \in \mathbb{C}$

System of numbers.

① Natural $\mathbb{N} = \{1, 2, 3, \cdots\}$

② Whole $\{0, 1, 2, 3, 4, \cdots\}$

③ Integer $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$

④ Rational number $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$

⑤ IRational number $\mathbb{Q}' = \{\sqrt{3}, \sqrt{2}, \pi$

⑥ Real numbers - $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$

⑦

$$x^2 + 1 = 0$$

$$x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$$

$$\sqrt{-1} = i$$

Complex numbers $\mathbb{C} : \{\underbrace{a}_{\text{real part}} + \underbrace{bi}_{\text{Imaginary part}}, a, b \in \mathbb{R}\}$

e.g $Z_1 = 2 + 3i$, $Z_2 = -2/3 + 3i\pi$

1) Addition/substraction

$Z_1 = a_1 + b_1 i$ and $Z_2 = a_2 + b_2 i$

$$Z_1 \mp Z_2 = (a_1 \mp a_2) + (b_1 \mp b_2) i$$

$\boxed{2}$ $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1$

i^3 i^2 i i

$\boxed{2}$ $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = 1$

$\boxed{3}$ Multiplication:

$$Z_1 * Z_2 = (a_1 + b_1 i) * (a_2 + b_2 i)$$
$$= a_1 a_2 + a_1 b_2 i + b_1 a_2 i - b_1 b_2$$
$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i$$

Ex ①: Perfor; for $Z_1 = \frac{1}{2} + 3i$ $Z_2 = -1 - \frac{4}{3} i$

① $Z_1 + Z_2$ ② $Z_1 - Z_2$ ③ $Z_1 * Z_2$

$= -\frac{1}{2} + \frac{5}{3} i$ $= 3/2 + \frac{13}{3} i$ $= (-1/2 + \frac{12}{3}) + (\frac{-4}{6} - 3) i$

$= \frac{21}{6} - \frac{22}{6} i$

Division:

Conjugate: $Z = a + bi \xrightarrow{\text{Conjugate}} \overline{Z} = a - bi$

e.g: $Z = -3i + 2$

$Z = a + bi$ $\Rightarrow \overline{Z} = 3i + 2$

$$Z * \overline{Z} = (a+bi)(a-bi) = a^2 + b^2$$

Norm of complex number Z denoted:

$$|Z| = \sqrt{Z * \overline{Z}}$$

Ex ②: for $Z = 3 + 4i$

Find $Z * \overline{Z}$ and $|Z|$

$Z * \overline{Z} = 3^2 + 4^2 = 25$

$\therefore |Z| = \sqrt{Z\overline{Z}} = 5$

Ex ③: Simplify $\dfrac{2-3i}{-4+i}$

Soln $\dfrac{2-3i}{-4+i} \times \dfrac{-4-i}{-4-i} = \dfrac{(-8-3)+(-2+12)i}{17}$

$(4)^2+(1)^2$

$= \dfrac{-11+10i}{17}$

Exer: Simplify $\dfrac{(2+i)(3-i)}{3i^2+2i-1}$

Proposition (**Fundamental Theorem of Algebra**). Every polynomial equation of one variable with complex coefficients has a complex solution.

$a^2 \mp 2ab + b^2$
$= (a \mp b)^2$

$x^2+2x+5=0$

$x^2+2x+1^2=-5+1^2$

$(x+1)^2=-4$

$x+1=\pm\sqrt{-4}=\pm 2i$

$x=-1\pm 2i$

$\frac{2}{2}=1$

$\sqrt{-4}=\sqrt{-1}\sqrt{4}$
$=2i$

Notation $z=a+bi$

$\Rightarrow z=(a,b)$

$a+0=a$

$a\times 1=a$

Definition: The set of complex numbers associated with addition and multiplication is defined as a **field** $(\mathbb{C}, +, \times)$

i. Addition is commutative and associative.
ii. Multiplication is commutative and associative.
iii. Addition has an identity: (0, 0).
iv. Multiplication has an identity: (1, 0).
v. Multiplication distributes with respect to addition.
vi. Subtraction (i.e., the inverse of addition) is defined everywhere.
vii. Division (i.e., the inverse of multiplication) is defined everywhere except when the divisor is zero.

$z_1+z_2=z_2+z_1$ | $z_1+(z_2+z_3)$
$=(z_1+z_2)+z_3$

$z_1 * z_2 = z_2 * z_1$

$0+0i$

$z_1+(0,0)=z_1$

$(a+bi)$

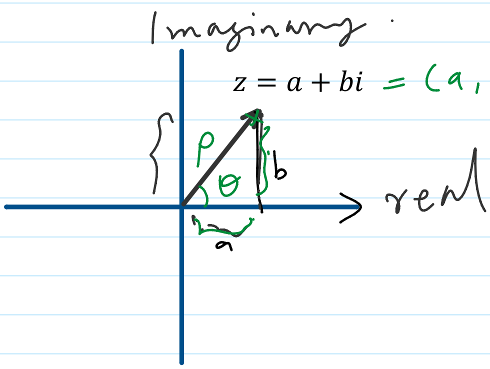
vii. Division (i.e., the inverse of multiplication) is defined everywhere except when the divisor is zero.

(iv) $(1,0) = 1 + 0i$ $\quad \overset{(a+bi)}{Z} * (1 + 0i) = Z$

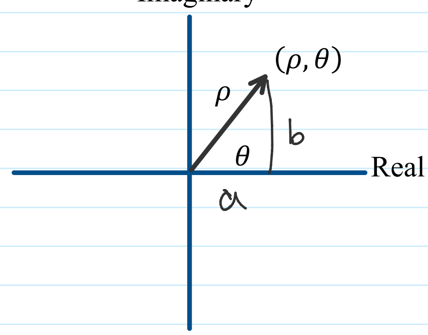
(v) $Z_1 * (Z_2 + Z_3) = Z_1 * Z_2 + Z_1 * Z_3$

(vi) Inverse of addition $\quad Z + (-Z) = 0 + 0i$

<u>The Geometric of Complex Numbers</u>



Complex Plane (Cartesian representation)



Complex Plane (Polar representation)

$$\frac{b}{a} = \frac{\cancel{\rho} \sin\theta}{\cancel{\rho} \cos\theta} = \tan\theta \leftarrow \begin{array}{l} a = \rho \cos\theta \\ b = \rho \sin\theta \end{array}$$

$$a^2 + b^2 = \rho^2 \cos^2\theta + \rho^2 \sin^2\theta = \rho^2 (\underbrace{\cos^2\theta + \sin^2\theta}_{1}) = \rho^2$$

$$\rho^2 = a^2 + b^2 \Rightarrow \rho = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad 0 \le \theta \le 2\pi$$

Example : Let $Z = 1 + i$. What is its polar representation?

$a = 1$, $b = 1$

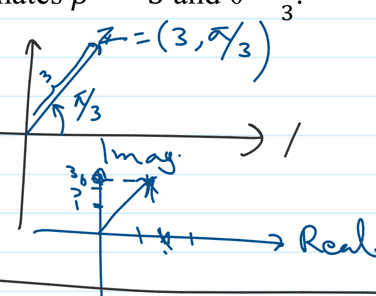
$$\rho = \sqrt{1^2 + 1^2} = \sqrt{2} \qquad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$Z = (\sqrt{2}, \pi/4)$$

Example : Draw the complex number given by the polar coordinates $\rho = 3$ and $\theta = \frac{\pi}{3}$. Compute its Cartesian coordinates.

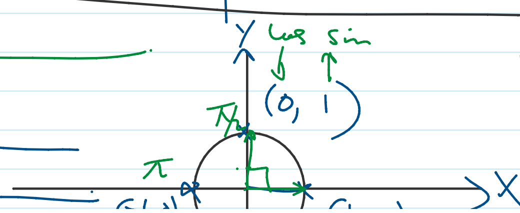
$$a = \rho \cos \pi/3 = 3 \cos \pi/3 = 3/2$$

$$b = \rho \sin \pi/3 = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$$



$$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

θ	$\cos\theta$	$\sin\theta$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
$30^\circ = \frac{\pi}{6}$	$\sqrt{3}/2$	$1/2$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$45^\circ = \pi/4$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}/2$	1

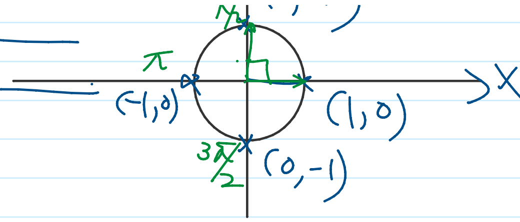


$a = \cancel{\rho} \cos\theta$

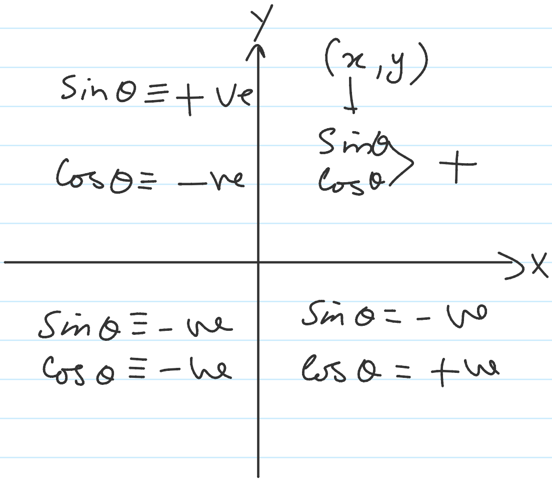
$x: a = \cos\theta$

$y: b = \sin\theta$

$45^0 = \pi/4$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}/2$	1
$60^0 = \pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$

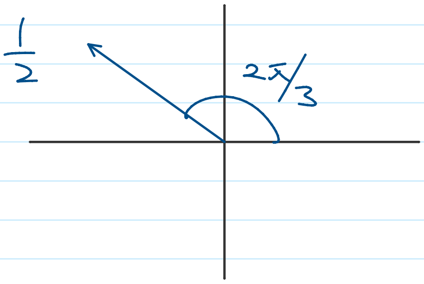


$y: b = \sin\theta$



$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(2\pi/3\right) = -\frac{1}{2}$



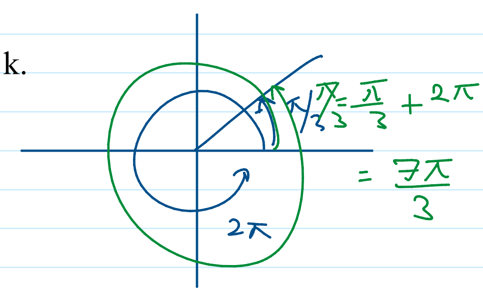
Definition : A complex number has a magnitude ρ and a phase θ.

Observe that a complex number has a unique polar representation only if we define the phase between 0 and 2π: $(0 \leq \theta \leq 2\pi \text{ and } \rho \geq 0)$

$\rho = \sqrt{a^2 + b^2}$

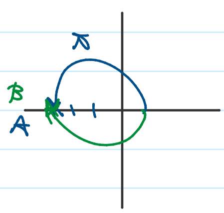
If θ is any thing

$\theta_1 = \theta_2$ if and only if $\theta_2 = \theta_1 + 2\pi k$, for some integer k.



Example : Are the numbers $(3, -\pi)$ and $(3, \pi)$ the same?

B A



$$Z = a + bi = \rho\cos\theta + i\rho\sin\theta$$
$$= \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$$

Euler's formula: $e^{i\theta} = cos(\theta) + isin(\theta)$

$$e^{i\pi/2} = \overset{=0}{\cos(\pi/2)} + i\overset{=1}{\sin(\pi/2)}$$
$$= i$$

$$e^{i\pi} = -1$$

Prove that $e^{i(\theta_1+\theta_2)} = e^{i\theta_1} \times e^{i\theta_2}$

$$e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$
$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$$
$$= \cos\theta_1(\cos\theta_2 + i\sin\theta_2) + \sin\theta_1(-\sin\theta_2 + i\cos\theta_2)$$

$= \cos\theta_1(\cos\theta_2 + i\sin\theta_2) + \sin\theta_1(-\sin\theta_2 + i\cos\theta_2)$

$(a^n)^m = a^{n\cdot m}$

$= \cos\theta_1 e^{i\theta_2} + i\sin\theta_1(i\sin\theta_2 + \cos\theta_2)$

$= \cos\theta_1 e^{i\theta_2} + i\sin\theta_1 e^{i\theta_2} = e^{i\theta_2} \times e^{i\theta_1}$

$$(e^{\theta i})^n = \cos(n\theta) + i\sin(n\theta).$$

$e^{n\theta i}$

Finally complex number can be written as $c = \rho e^{i\theta}$

e.x: $e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}(1+i)$ $\quad = \frac{1}{\sqrt{2}}$ $\quad |0\rangle \quad |1\rangle$

Given two complex numbers in polar coordinates, $z_1 = (\rho_1, \theta_1)$ and $z_2 = (\rho_2, \theta_2)$, their product can be obtained by simply multiplying their magnitude and adding their phase:

$\rho_1 e^{i\theta_1}\rho_2 e^{i\theta_2} = \rho_1\rho_2 e^{i(\theta_1+\theta_2)}$

$$z_1 \times z_2 = (\rho_1 \times \rho_2, \theta_1 + \theta_2)$$

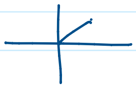
$\boxed{a^n \cdot a^m = a^{n+m}}$

$Z = a + bi$

$\rho = \sqrt{a^2+b^2}$

$\theta - \tan^{-1}\left(\frac{b}{\ }\right)$

Example : Let $z_1 = 1 + i$ and $z_2 = -1 + i$. Find their product according to the algebraic rule.



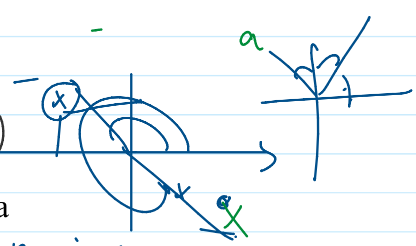
$\rho_1 = \sqrt{2}$ $\quad \rho_2 = \sqrt{2}$

$\theta_1 = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$ $\quad \theta_2 = \tan^{-1}\left(\frac{1}{-1}\right) = 3\pi/4$

$Z_1 \times Z_2 = (\rho_1 \cdot \rho_2, \theta_1 + \theta_2) = (2, \pi)$

If $z_2 = (\rho_1, \theta_1)$ and $z_2 = (\rho_2, \theta_2)$, what is $\frac{z_1}{z_2}$

$\frac{Z_1}{Z_2} = \frac{\rho_1 e^{i\theta_1}}{\rho_2 e^{i\theta_2}} = \frac{\rho_1}{\rho_2} e^{i(\theta_1-\theta_2)} = \left(\frac{\rho_1}{\rho_2}, \theta_1 - \theta_2\right)$



Generalized nth Power : If $z = (\rho, \theta)$ is a complex number in polar form and n a positive integer, its nth power is just $z^n = (\rho^n, n\theta)$.

$Z^n = (\rho e^{i\theta})^n = \rho^n e^{in\theta} = (\rho^n, n\theta)$

Example: Let $z = 1 - i$. Calculate its fifth power, and revert the answers to Cartesian coordinates.

$\rho = \sqrt{2}$, $\theta = \tan^{-1}(-1/1) = \frac{7\pi}{4}$, $\therefore Z^5 = \left(\sqrt{2}^5, 5\cdot\frac{7\pi}{4}\right) = \left(4\sqrt{2}, \frac{35\pi}{4}\right)$

Generalized nth root : If $z = (\rho, \theta)$ is a complex number in polar form and n a positive integer, its nth root is just $z^{1/n} = (\rho^{1/n}, \frac{1}{n}\theta)$

Example : Find all the cube roots of $z = 1 + i$.