1.1 Revision of Complex Numbers

Monday, August 25, 2025 8:26 PM

"*Associated to any isolated physical system is a complex vector space with inner product (i.e. a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.*" (Nielsen and Chuang).

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$\alpha, \beta \in \mathbb{C}$

System of numbers

1. Natural $\mathbb{N} = \{1, 2, 3, \ldots\}$
2. Whole $\{0, 1, 2, 3, 4, \ldots\}$
3. Integer $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
4. Rational number $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0\}$
5. IRational number $\mathbb{Q}' = \{\sqrt{3}, \sqrt{2}, \pi$
6. Real numbers - $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$
7.

$$x^2 + 1 = 0$$

$$x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$$

$$\sqrt{-1} = i$$

Complex numbers $\mathbb{C} : \{a + bi, a, b \in \mathbb{R}\}$

a → real part; bi → Imaginary part.

e.g $Z_1 = 2 + 3i$, $Z_2 = -2/3 + 3i\pi$

1) Addition/substraction

$$Z_1 = a_1 + b_1 i \quad \text{and} \quad Z_2 = a_2 + b_2 i$$

$$Z_1 \mp Z_2 = (a_1 \mp a_2) + (b_1 \mp b_2) i$$

2) $i^2 = i \, i = \sqrt{-1} \cdot \sqrt{-1} = -1$

$i^3 = i^2 \, i \ldots$

$\boxed{2}$ $i^2 = i \cdot i = \sqrt{-1} \cdot \sqrt{-1} = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = 1$

$\boxed{3}$ Multiplication:

$$Z_1 * Z_2 = (a_1 + b_1 i) * (a_2 + b_2 i)$$
$$= a_1 a_2 + a_1 b_2 i + b_1 a_2 i - b_1 b_2$$
$$= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i$$

Ex ①: Perfor; for $Z_1 = \frac{1}{2} + 3i$ $Z_2 = -1 - \frac{4}{3} i$

① $Z_1 + Z_2$ ② $Z_1 - Z_2$ ③ $Z_1 * Z_2$

$= -\frac{1}{2} + \frac{5}{3} i$ $= 3/2 + \frac{13}{3} i$ $= \left(-\frac{1}{2} + \frac{12}{3}\right) + \left(\frac{-4}{6} - 3\right) i$

$= \frac{21}{6} - \frac{22}{6} i$

Division:

Conjugate: $Z = a + bi \xrightarrow{\text{Conjugate}} \overline{Z} = a - bi$

e.g: $Z = -3i + 2$

$Z = a + bi$ $\Rightarrow \overline{Z} = 3i + 2$

$$Z * \overline{Z} = (a + bi)(a - bi) = a^2 + b^2$$

Norm of complex number Z denoted:

$$|Z| = \sqrt{Z * \overline{Z}}$$

E.X ②: for $Z = 3 + 4i$

Find $Z * \overline{Z}$ and $|Z|$

$Z * \overline{Z} = 3^2 + 4^2 = 25$

$\therefore |Z| = \sqrt{Z\overline{Z}} = 5$

Ex ③: Simplify $\frac{2-3i}{-4+i}$

Soln $\frac{2-3i}{-4+i} \times \frac{-4-i}{-4-i} = \frac{(-8-3)+(-2+12)i}{17}$

$(4)^2+(1)^2$

$= \frac{-11+10i}{17}$

Exer: Simplify $\frac{(2+i)(3-i)}{3i^2+2i-1}$

Proposition (**Fundamental Theorem of Algebra**). Every polynomial equation of one variable with complex coefficients has a complex solution.

$\frac{2x^2}{2}+\frac{2x}{2}+\frac{4}{2}=\frac{0}{2}$

$x^2+x+2=0$

$x^2+x+\left(\frac{1}{2}\right)^2=-2+\left(\frac{1}{2}\right)^2$

$\left(\frac{1}{2}\right)^2$

$\left(x+\frac{1}{2}\right)^2=-\frac{3}{4}$

$\boxed{x=\pm\frac{\sqrt{3}}{2}i-\frac{1}{2}}$

$\pm\sqrt{-3/4}$

$=\pm\frac{\sqrt{3}\,i}{2}$

$x^2+2x+5=0$

$x^2+2x+1^2=-5+1^2$

$\frac{2}{2}=1$

$(x+1)^2=-4$

$\sqrt{-4}=\sqrt{-1}\sqrt{4}$

$=2i$

$x+1=\pm\sqrt{-4}=\pm 2i$

$x=-1\pm 2i$

Notation $z=a+bi$

$\Rightarrow z=(a,b)$

$a+0=a$

$a\times 1=a$

Definition: The set of complex numbers associated with addition and multiplication is defined as a **field** $(\mathbb{C},+,\times)$

i. Addition is commutative and associative.
ii. Multiplication is commutative and associative.
iii. Addition has an identity: (0, 0).
iv. Multiplication has an identity: (1, 0).
v. Multiplication distributes with respect to addition.
vi. Subtraction (i.e., the inverse of addition) is defined everywhere.
vii. Division (i.e., the inverse of multiplication) is defined everywhere except when the divisor is zero.

$z_1+z_2=z_2+z_1$ | $z_1+(z_2+z_3)$

$z_1 * z_2=z_2 * z_1$ $=(z_1+z_2)+z_3$

$z_1+\overset{0+0i}{(0,0)}=z_1$

(a,b)

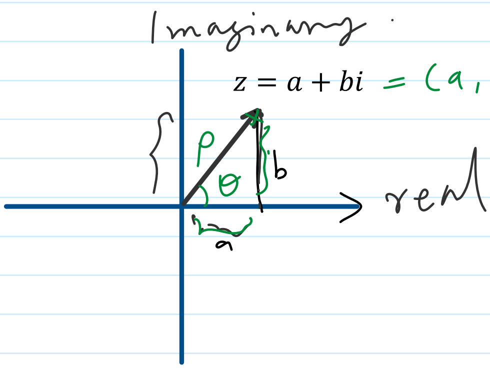
vii. Division (i.e., the inverse of multiplication) is defined everywhere except when the divisor is zero.

(iv) $(1,0) = 1+0i \qquad Z*(1+0i) = Z$ (a+b)

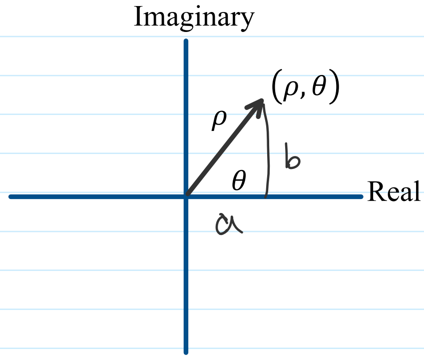
(v) $Z_1 * (Z_2 + Z_3) = Z_1 * Z_2 + Z_1 * Z_3$

(vi) Inverse of addition $\quad Z + (-Z) = 0 + 0i$

<u>The Geometric of Complex Numbers</u>



Complex Plane (Cartesian representation)



Complex Plane (Polar representation)

$$\frac{b}{a} = \frac{\cancel{\rho}\sin\theta}{\cancel{\rho}\cos\theta} = \tan\theta \leftarrow \quad a = \rho\cos\theta, \quad b = \rho\sin\theta$$

$$a^2 + b^2 = \rho^2\cos^2\theta + \rho^2\sin^2\theta = \rho^2\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = \rho^2$$

$$\rho^2 = a^2 + b^2 \Rightarrow \rho = \sqrt{a^2+b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad 0 \le \theta \le 2\pi$$

Example : Let $Z = 1 + i$. What is its polar representation?

$a = 1$, $b = 1$

$$\rho = \sqrt{1^2+1^2} = \sqrt{2} \qquad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

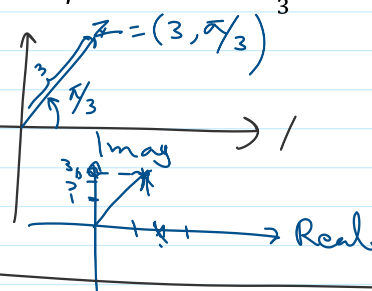
$$Z = (\sqrt{2}, \pi/4)$$

Example : Draw the complex number given by the polar coordinates $\rho = 3$ and $\theta = \frac{\pi}{3}$. Compute its Cartesian coordinates.

$$a = \rho\cos\pi/3 = 3\cos\pi/3 = 3/2$$

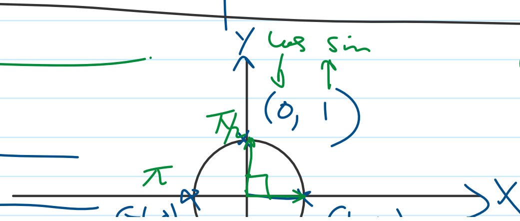
$$b = \rho\sin\pi/3 = 3\left(\frac{\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{2}$$

$$\frac{\frac{1}{\cancel{2}}}{\frac{\sqrt{3}}{\cancel{2}}} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$



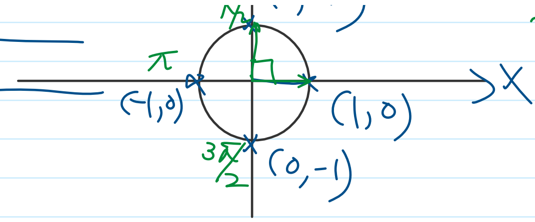
$\frac{\sin\theta}{\cos\theta}$

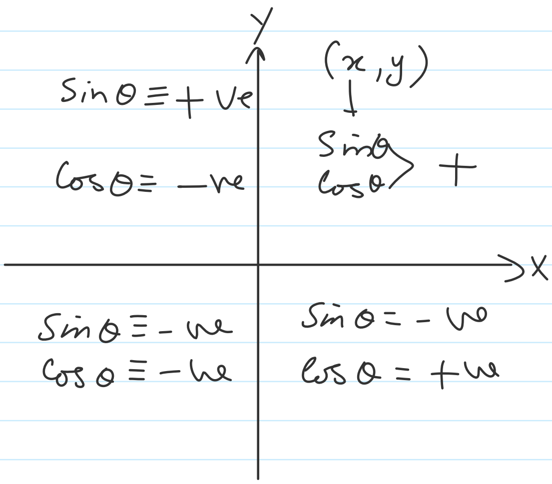
θ \ π rad	$\cos\theta$	$\sin\theta$	$\tan\theta$
$30^\circ = \frac{\pi}{6}$	$\sqrt{3}/2$	$1/2$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$45^\circ = \pi/4$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}/2$	1



$a = \cancel{\rho}\cos\theta$
$a = \cos\theta$
$b = \sin\theta$

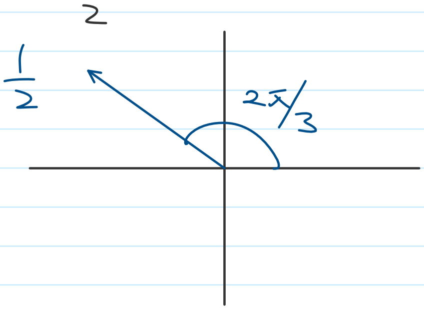
$45^\circ = \pi/4$	$\frac{\sqrt{2}}{2}$	$\sqrt{2}/2$	1
$60^\circ = \pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}$





$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos(2\pi/3) = -\frac{1}{2}$



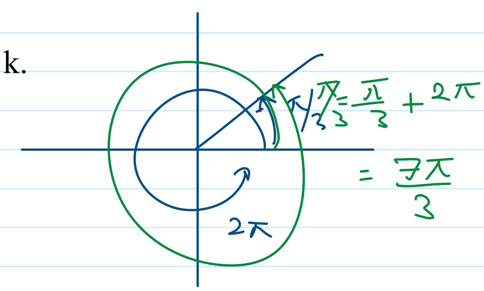
Definition : A complex number has a magnitude ρ and a phase θ.

Observe that a complex number has a unique polar representation only if we define the phase between 0 and 2π: $(0 \le \theta \le 2\pi \text{ and } \rho \ge 0)$

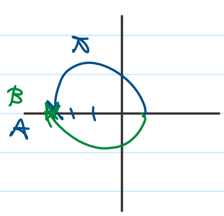
$\rho = \sqrt{a^2 + b^2}$

If θ is any thing

$$\theta_1 = \theta_2 \text{ if and only if } \theta_2 = \theta_1 + 2\pi k, \text{ for some integer k.}$$



Example : Are the numbers $\overset{B}{(3,-\pi)}$ and $\overset{A}{(3, \pi)}$ the same?



$$Z = a + bi = \rho\cos\theta + \rho i \sin\theta$$
$$= \rho\boxed{(\cos\theta + i\sin\theta)} = \rho e^{i\theta} \Longrightarrow Z = \rho e^{i\theta}$$

Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

$e^{i\pi/2} = \overset{=0}{\cos(\pi/2)} + i\overset{=1}{\sin(\pi/2)}$
$= i$

$e^{i\pi} = -1$

Prove that $e^{i(\theta_1+\theta_2)} = e^{i\theta_1} \times e^{i\theta_2}$

$e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$

$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)$

$= \cos\theta_1(\cos\theta_2 + i\sin\theta_2) + \sin\theta_1(-\sin\theta_2 + i\cos\theta_2)$

$$= \cos\theta_1(\cos\theta_2 + i\sin\theta_2) + \sin\theta_1(-\sin\theta_2 + i\cos\theta_2)$$

$(a^n)^m = a^{n\cdot m}$

$$= \cos\theta_1\, e^{i\theta_2} + i\sin\theta_1\,(i\sin\theta_2 + \cos\theta_2)$$

$$= \cos\theta_1\, e^{i\theta_2} + i\sin\theta_1\, e^{i\theta_2} = e^{i\theta_2} \times e^{i\theta_1}$$

$$(e^{\theta i})^n = \cos(n\theta) + i\sin(n\theta).$$

$e^{n\theta i}$

Finally complex number can be written as $c = \rho e^{i\theta}$

e.x: $e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}(1+i)$; $= \frac{1}{\sqrt{2}}$; $|0\rangle$ $|1\rangle$

Given two complex numbers in polar coordinates, $z_1 = (\rho_1, \theta_1)$ and $z_2 = (\rho_2, \theta_2)$, their product can be obtained by simply multiplying their magnitude and adding their phase:

$\rho_1 e^{i\theta_1}\, \rho_2 e^{i\theta_2} = \rho_1\rho_2\, e^{i(\theta_1+\theta_2)}$

$$z_1 \times z_2 = (\rho_1 \times \rho_2, \theta_1 + \theta_2)$$

$\boxed{a^n \cdot a^m = a^{n+m}}$

Example : Let $z_1 = 1 + i$ and $z_2 = -1 + i$. Find their product according to the algebraic rule.

$Z = a + bi$

$\rho = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

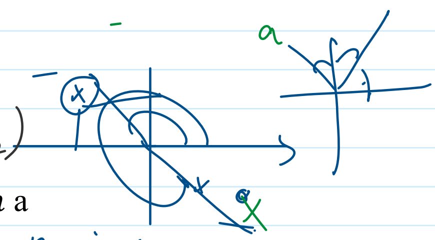
$\rho_1 = \sqrt{2}$ $\quad$ $\rho_2 = \sqrt{2}$

$\theta_1 = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$ $\quad$ $\theta_2 = \tan^{-1}\left(\frac{1}{-1}\right) = 3\pi/4$

$$Z_1 \times Z_2 = (\rho_1\rho_2, \theta_1+\theta_2) = (2, \pi)$$

If $z_2 = (\rho_1, \theta_1)$ and $z_2 = (\rho_2, \theta_2)$, what is $\frac{z_1}{z_2}$

$$\frac{Z_1}{Z_2} = \frac{\rho_1 e^{i\theta_1}}{\rho_2 e^{i\theta_2}} = \frac{\rho_1}{\rho_2} e^{i(\theta_1-\theta_2)} = \left(\frac{\rho_1}{\rho_2}, \theta_1 - \theta_2\right)$$



Generalized nth Power : If $z = (\rho, \theta)$ is a complex number in polar form and n a positive integer, its nth power is just $z^n = (\rho^n, n\theta)$

$$Z^n = (\rho e^{i\theta})^n = \rho^n e^{in\theta} = (\rho^n, n\theta)$$

Example: Let $z = 1 - i$. Calculate its fifth power, and revert the answers to Cartesian coordinates.

$\rho = \sqrt{2}$, $\theta = \tan^{-1}(-1/1) = \frac{7\pi}{4}$, $\therefore Z^5 = \left(\sqrt{2}^5, 5\cdot\frac{7\pi}{4}\right) = \left(4\sqrt{2}, \frac{35\pi}{4}\right)$

Generalized nth root : If $z = (\rho, \theta)$ is a complex number in polar form and n a positive integer, its nth root is just $z^{1/n} = (\rho^{1/n}, \frac{1}{n}\theta)$

$Z = \rho e^{i\theta}$

$Z^{1/n} = (\rho e^{i\theta})^{\frac{1}{n}} = \rho^{\frac{1}{n}} \cdot e^{\frac{i\theta}{n}} \Rightarrow Z = (\rho, \theta)$

Example : Find all the cube roots of $z = 1 + i$.

Cube

$\rho = \sqrt{1^2+1^2} = \sqrt{2}$ $\quad$ $Z = a + bi \Rightarrow (a, b)$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$ $\quad$ $\sqrt{8}$ $\quad$ $(\sqrt{2})^{\frac{1}{3}} = \left(2^{\frac{1}{2}}\right)^{\frac{1}{3}}$

$e^{i3\pi/}$ $\quad$ $e^{i3\pi/}$

Cube

$$\left(e^{i\frac{\pi}{4}}\right)^{\frac{1}{3}}$$

$$e^{i\frac{1}{3}\left(\frac{\pi}{4}\right)}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$\sqrt{8}$$

$$z^3 = r^3 \cdot e^{3i\theta} = (\sqrt{2})^3 e^{i3\pi/4} = 2\sqrt{2}\, e^{i3\pi/4}$$

$$z^{1/3} = (\sqrt{2})^{1/3} e^{i\pi/12} = \sqrt[6]{2}\, e^{i\pi/12}$$

$$(\sqrt{2})^{\frac{1}{3}} = \left(2^{\frac{1}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{6}} = \sqrt[6]{2}$$

1.2 Complex Vector Space

Sunday, August 31, 2025 6:30 PM

Representation of Vectors over Complex Numbers

$$V \in \mathbb{C}^n \qquad V = (v_1, v_2, v_3, \ldots v_n)$$

$\mathbb{V} = \mathbb{C}^2$

$u = (1+i, 3) \quad v = (1-4i, 2+i)$

$u + v = (2-3i, 5+i)$

Definition 2.2.1: A vector space $\mathbb{V} = \mathbb{C}^n$ over $\mathbb{C}$ is a set $\mathbb{V}$ with two operations:

- Vector addition: for every $v, u \in \mathbb{V}$, there is an element $v + u \in \mathbb{V}$
- Scalar multiplication: for every $c \in \mathbb{C}$ and $v \in \mathbb{V}$, there is an element $c \cdot v \in \mathbb{V}$ such that the following axioms hold: $c = (1+i) \quad v = (1-4i, 2+i) \quad cv \in \mathbb{C}^2$

I. $v + u = u + v$ for every $v, u \in \mathbb{V}$ (commutativity of addition)

II. $(v + u) + w = v + (u + w)$ for every $v, u, w \in \mathbb{V}$ (associativity of addition)

$v = (1-4i, 2+i) + (0,0) = v$

III. There is an element $\mathbf{0} \in \mathbb{V}$ such that $v + \mathbf{0} = v$ for every $v \in \mathbb{V}$ ($\mathbf{0}$ is vector additive identity)

IV. For every $v \in \mathbb{V}$ there is -$v \in \mathbb{V}$ such that $v + (-v) = \mathbf{0}$ (vector additive inverse) $\Rightarrow v + (-v) = 0$

V. $1 \cdot v = v$ for every $v \in \mathbb{V}$ (where 1 is multiplicative identity of $\mathbb{C}$) $1 \cdot v = v$

VI. $c_1 \cdot (c_2 \cdot v) = (c_1 \cdot c_2) \cdot v$ for every $c_1, c_2 \in \mathbb{C}$ and $v \in \mathbb{V}$ (associativity of scalar multiplication

VII. $c_1 \cdot (u + v) = c_1 \cdot u + c_1 \cdot v$ for every $c_1 \in \mathbb{C}$ and $v, u \in \mathbb{V}$ (distributivity)

VIII. $(c_1 + c_2) \cdot v = c_1 \cdot v + c_2 \cdot v$ for every $c_1, c_2 \in \mathbb{C}$ and $v \in \mathbb{V}$ (distributivity)

no. of rows

no. of

Note: $\mathbb{C}^{m \times n}$, the set of all m-by-n matrices with complex entries,

e.g. $A \in \mathbb{C}^{2 \times 3}$

$$A = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & & c_{mn} \end{bmatrix}, \quad A = \begin{bmatrix} 1+i & 2 & 0 \\ \frac{1}{2}+2i & -\frac{3}{4}-\frac{5}{2}i & 5 \end{bmatrix}$$

Exercise 2.2.3 Let $c_1 = 2i$, $c_2 = 1 + 2i$, and $A = \begin{bmatrix} 1-i & 3 \\ 2+2i & 4+i \end{bmatrix}$. Verify Properties (vi) and (viii) in showing $\mathbb{C}^{2\times 2}$ is a complex vector space.

Type equation here.

$$(c_1 c_2) A = c_1 (c_2 A)$$

$1+2i$

$$c_2 A = \begin{bmatrix} 3+i & 3+6i \\ -2+6i & 2+9i \end{bmatrix}$$

$2+2i$

$1+2i$

$1-1i$

$\times \; 1+2i$

$(1+2) + i$

Definition 2.2.2: Let A $\in \mathbb{C}^{m\times n}$, the **transpose** of A is denoted by A^T and is defined as $A^T[i, j] = A[j, i]$

Definition 2.2.3: Let A $\in \mathbb{C}^{m\times n}$, the **Conjugate** of A is denoted by $\overline{A}$ and is defined as $\overline{A}[i, j] = \overline{A[i, j]}$

Definition 2.2.4: Let A $\in \mathbb{C}^{m\times n}$, the **dagger or adjoint** of A is denoted by $A^\dagger$ and is defined as $A^\dagger = \overline{A[j, i]}$

Example 2.4: Find the transpose, conjugate, and adjoint of

$A_{n\times n} = A[i, j] \quad i = 1 \to m, \; j = 1 \to n$

$$A = \begin{bmatrix} 6-3i & 2+12i & -19i \\ 0 & 5+2.1i & 17 \\ 1 & 2+5i & 3-4.5i \end{bmatrix}.$$

1) $A^T_{n\times m} = A[j, i]$

$$A^T = \begin{bmatrix} 6-3i & 0 & 1 \\ 2+12i & 5+2.1i & 2+5i \end{bmatrix}$$

2) $\overline{A} = \overline{A[i, j]}$

$$A^T = \begin{bmatrix} 6-3i & 0 & 1 \\ 2+12i & 5+2.1i & 2+5i \\ -19i & 17 & 3-4.5i \end{bmatrix}$$

$$\overline{A} = \begin{bmatrix} 6+3i & 2-12i & +19i \\ 0 & 5-2.1i & 17 \\ 1 & 2-5i & 3+4.5i \end{bmatrix}$$

$$A^{+} = \overline{(A^T)} = \begin{bmatrix} 6+3i & 0 & 1 \\ 2-12i & 5-2.1i & 2-5i \\ +19i & 17 & 3+4.5i \end{bmatrix}$$

2) $\overline{A} = \overline{A[i,j]}$

③ $A^{+} = \overline{(A^T)} = \overline{A[j,i]}$

These operations satisfy the following properties for all $c \in \mathbb{C}$ and for all A, $B \in \mathbb{C}^{m\times n}$:

(i) Transpose is idempotent: $(A^T)^T = A$.
(ii) Transpose respects addition: $(A+B)^T = A^T + B^T$.
(iii) Transpose respects scalar multiplication: $(c \cdot A)^T = c \cdot A^T$.

(iv) Conjugate is idempotent: $\overline{\overline{A}} = A$.
(v) Conjugate respects addition: $\overline{A+B} = \overline{A} + \overline{B}$.
(vi) Conjugate respects scalar multiplication: $\overline{c \cdot A} = \overline{c} \cdot \overline{A}$.
(vii) Adjoint is idempotent: $(A^\dagger)^\dagger = A$.
(viii) Adjoint respects addition: $(A+B)^\dagger = A^\dagger + B^\dagger$.
(ix) Adjoint relates to scalar multiplication: $(c \cdot A)^\dagger = \overline{c} \cdot A^\dagger$.

$$(\overline{c \cdot A})^T = (\overline{c\, A[i,j]})^T = (\overline{c}\ \overline{A[i,j]})^T = \overline{c}\ \overline{A[j,i]} = \overline{c}\ A^{+}$$

$$(c \cdot A)^{+} = (\overline{c \cdot A})^T = (\overline{c} \cdot \overline{A})^T = \overline{c} \cdot A^{+}$$

Matrix Multiplication: $\mathbb{C}^{m\times n} \times \mathbb{C}^{n\times p} \longrightarrow \mathbb{C}^{m\times p}$.

$A_{m\times n} \times B_{n\times p} = C_{m\times p}$

Formally, given A in $\mathbb{C}^{m\times n}$ and B in $\mathbb{C}^{n\times p}$, we construct $A \star B$ in $\mathbb{C}^{m\times p}$ as

$$C[j,k] = (A \star B)[j,k] = \sum_{h=0}^{n-1} (A[j,h] \times B[h,k]).$$

j row of A

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 5 & 7 \end{bmatrix}_{2\times 3} \qquad B = {}_{3\times 1}\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$A \times B\,[1,1]$
$= A[1,1]\,B[1,1] + A[1,2]\,B[2,1] + A[1,3]\,B[3,1]$

$$= \begin{bmatrix} 2\times 0 + 3\times -1 + 6\times 3 \\ 4\times 0 + 5\times -1 + 7\times 3 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 15 \\ 16 \end{bmatrix}$$

For every n, there is a special n-by-n matrix called the **identity matrix**,

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \cdot I = A$$

Matrices multiplication properties: For all $A, B, C \in \mathbb{C}^{n\times n}$,

(i) Matrix multiplication is associative: $(A \star B) \star C = A \star (B \star C)$.
(ii) Matrix multiplication has I_n as a unit: $I_n \star A = A = A \star I_n$.
(iii) Matrix multiplication distributes over addition:

$$A \star (B + C) = (A \star B) + (A \star C),$$

$$(B + C) \star A = (B \star A) + (C \star A).$$

(iv) Matrix multiplication respects scalar multiplication:

$$c \cdot (A \star B) = (c \cdot A) \star B = A \star (c \cdot B).$$

(v) Matrix multiplication relates to the transpose:

$$(A \star B)^T = B^T \star A^T.$$

(vi) Matrix multiplication respects the conjugate:

$$\overline{A \star B} = \overline{A} \star \overline{B}.$$

(vii) Matrix multiplication relates to the adjoint:

$$(A \star B)^\dagger = B^\dagger \star A^\dagger.$$

$$\left(\overline{(A * B)}\right)^T = (\overline{A} * \overline{B})^T = \overline{B}^T * \overline{A}^T = B^\dagger * A^\dagger$$

$$(A_{2\times3} * B_{3\times4})^T = (D_{2\times4})^T = C_{4\times2}$$

$$A^T_{3\times2} \quad B^T_{4\times3}$$

$$B^T_{4\times3} * A^T_{3\times2} = C_{4\times2}$$

Exercise 2.5: Given A amd B, show that $(A * B)^\dagger = B^\dagger * A^\dagger$

$$A = \begin{bmatrix} 3+2i & 0 & 5-6i \\ 1 & 4+2i & i \\ 4-i & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2-i & 6-4i \\ 0 & 4+5i & 2 \\ 7-4i & 2+7i & 0 \end{bmatrix}.$$

$$A^\dagger = \begin{bmatrix} 3-2i & 1 & 4+i \\ 0 & 4-2i & 0 \\ 5+6i & -i & 4 \end{bmatrix}$$

Definition 2.2.4 *Given two complex vector spaces $\mathbb{V}$ and $\mathbb{V}'$, we say that $\mathbb{V}$ is a* **complex subspace** *of $\mathbb{V}'$ if $\mathbb{V}$ is a subset of $\mathbb{V}'$ and the operations of $\mathbb{V}$ are restrictions of operations of $\mathbb{V}'$.*

$S \{0, \ldots, 9\}$

$S' = \{1, 2, 3\}$

Equivalently, $\mathbb{V}$ is a complex subspace of $\mathbb{V}'$ if $\mathbb{V}$ is a subset of the set $\mathbb{V}'$ and

(i) $\mathbb{V}$ is closed under addition: For all V_1 and V_2 in $\mathbb{V}$, $V_1 + V_2 \in \mathbb{V}$.
(ii) $\mathbb{V}$ is closed under scalar multiplication: For all $c \in \mathbb{C}$ and $V \in \mathbb{V}$, $c \cdot V \in \mathbb{V}$.

$\mathbb{V} = (V, +, *)$ subspace

$\mathbb{V}' = (V', +, *)$

Example 2.2.7: Consider the set of all vectors of $\mathbb{C}^9$ with the second, fifth, and eighth position elements being 0:

$V \in \mathbb{C}^9 \quad V = (v_1, 0, v_3, v_4, 0, v_6, v_7, 0, v_9) \qquad \mathbb{V} \subseteq \mathbb{V}'$

$\mathbb{C}^9 \qquad v_i \in \mathbb{C}$

$$p(x) = v_1 + v_3x^2 + v_4x^3 + v_6x^5 + v_7x^6 + v_9x^8$$

Definition 2.2.5: The set of polynomials of degree n or less in one variable with coefficients in $\mathbb{C}$ is form a complex vector space.

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

Definition 2.2.5 *Let $\mathbb{V}$ and $\mathbb{V}'$ be two complex vector spaces. A* **linear map** *from $\mathbb{V}$ to $\mathbb{V}'$ is a function $f : \mathbb{V} \longrightarrow \mathbb{V}'$ such that for all $V, V_1, V_2 \in \mathbb{V}$, and $c \in \mathbb{C}$,*

$V_1, V_2 \in V$

(i) *f respects the addition: $f(V_1 + V_2) = f(V_1) + f(V_2)$,*
(ii) *f respects the scalar multiplication: $f(c \cdot V) = c \cdot f(V)$.*

Definition 2.2.6 *Two complex vector spaces $\mathbb{V}$ and $\mathbb{V}'$ are* **isomorphic** *if there is a one-to-one onto linear map $f : \mathbb{V} \longrightarrow \mathbb{V}'$. Such a map is called an* **isomorphism**.

Example 2.2.8: Show that the map $f: \mathbb{C} \longrightarrow \mathbb{R}^{2\times 2}$ define below is an isomorphic from $\mathbb{C}$ to $\mathbb{R}^{2\times 2}$

$$f(x+iy) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

$V_1 = x_1 + iy_1$

$V_2 = x_2 + iy_2$

$$f(V_1+V_2) = f((x_1+x_2) + i(y_1+y_2))$$

$$= \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ -y_1-y_2 & x_1+x_2 \end{bmatrix} \stackrel{?}{=} f(V_1) + f(V_2)$$

1.3 Basis and Dimension

Monday, September 2, 2024 8:23 AM

Basis and Dimension

Definition 2.3.1 *Let $\mathbb{V}$ be a complex (real) vector space. $V \in \mathbb{V}$ is a* **linear combination** *of the vectors $V_0, V_1, \ldots, V_{n-1}$ in $\mathbb{V}$ if V can be written as*

$$V = \underline{c_0} \cdot \underline{V_0} + \underline{c_1} \cdot \underline{V_1} + \cdots + \underline{c_{n-1}} \cdot \underline{V_{n-1}} \quad \text{for some } c_0, c_1, \ldots, c_{n-1} \text{ in } \mathbb{C}\ (\mathbb{R}).$$

$= v$

Example 1: construct a vector $v \in \mathbb{V} = \mathbb{C}^3$ as a linear combination of the following vectors:

$$v_1 = \begin{bmatrix} i \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2+i \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$
$$= v_1 + v_2 + v_3 = \begin{bmatrix} 4+i \\ 2+i \\ 3 \end{bmatrix}$$

Definition 2.3.2 *A set $\{V_0, V_1, \ldots, V_{n-1}\}$ of vectors in $\mathbb{V}$ is called* **linearly independent** *if*

$$\mathbf{0} = c_0 \cdot V_0 + c_1 \cdot V_1 + \cdots + c_{n-1} \cdot V_{n-1}$$

implies that $c_0 = c_1 = \cdots = c_{n-1} = 0$. This means that the only way that a linear combination of the vectors can be the zero vector is if all the c_j are zero.

$2^1 + 2^1 = 2$

$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{C}^4$, $|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $|10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $|11\rangle$

Example 2: Show that the following vectors are linearly independent

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \to 0 \\ \to 1 \\ \to 2 \\ \to 3 \end{matrix}$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\begin{bmatrix} c_1 \\ c_1 + c_2 \\ c_1 + c_2 + c_3 \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{c_1 = 0}$$

$c_1 + c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

$c_1 + c_2 + c_3 = 0 \Rightarrow \boxed{c_3 = 0}$

Example 3: Show that the following vectors are not linearly independent (**linear dependent**)

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\begin{bmatrix} c_1 + 2c_3 \\ c_1 + c_2 - c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$c_1 + 2c_3 = 0 \Rightarrow c_1 = -2c_3$

$c_1 + c_2 - c_3 = 0$ if $\boxed{c_3 = 1} \Rightarrow \boxed{c_1 = -2}$

$$\begin{bmatrix} c_1 + 2c_3 \\ c_1 + c_2 - c_3 \\ c_1 + c_2 - c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{array}{l} c_1 + 2c_3 = 0 \Rightarrow c_1 = -2c_3 \\ c_1 + c_2 - c_3 = 0 \quad \text{if } \boxed{c_3 = 1} \Rightarrow \boxed{c_1 = -2} \\ c_1 + c_2 - c_3 = 0 \qquad \boxed{c_2 = -3} \end{array}$$

Definition 2.3.3 *A set $\mathcal{B} = \{V_0, V_1, \ldots, V_{n-1}\} \subseteq \mathbb{V}$* ($\mathbb{C}^n$) *of vectors is called a* **basis** *of a (complex) vector space $\mathbb{V}$ if both*

(i) *every, $V \in \mathbb{V}$ can be written as a linear combination of vectors from $\mathcal{B}$ and*

$V = c_0 V_0 + c_1 V_1 + \cdots + c_{n-1} V_{n-1} = \sum_{i=0}^{n-1} c_i V_i$

(ii) *$\mathcal{B}$ is linearly independent.* $\Rightarrow \forall c_i \quad c_0 V_0 + c_1 V_1 + c_2 V_2 + \cdots + c_{n-1} V_{n-1} = 0$

$\Rightarrow c_i = 0$

Standard Basis

■ $\mathbb{C}^n$ (and $\mathbb{R}^n$):

c_0

$$E_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad E_i = \begin{array}{r} \\ \\ i \rightarrow \\ \\ \end{array}\begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \ldots, \quad E_{n-1} = \begin{array}{r} \\ \\ \\ n-1 \rightarrow \end{array}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Note: Every vector $v = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{V} = \mathbb{C}^n$ can be written as follows

$$v = \sum_{i=0}^{n-1} c_i E_i = c_0 E_0 + c_1 E_1 + \cdots + c_{n-1} E_{n-1}$$

$$= \begin{bmatrix} c_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

The basis for the vector space $\mathbb{C}^{m \times n}$ consists of matrices of the form

$$E_{j,k} = \begin{array}{c|cccccc} & \mathbf{0} & \mathbf{1} & \cdots & \boldsymbol{k} & \cdots & \boldsymbol{n-1} \\ \hline \mathbf{0} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \mathbf{1} & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & & \cdots & \vdots \\ \boldsymbol{j} & 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & & \cdots & \vdots \\ \boldsymbol{m-1} & 0 & 0 & \cdots & 0 & \cdots & 0 \end{array},$$

k column   j row

$$E_{0,0} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

Any m-by-n matrix, A can be written as the sum:

$$A = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \underbrace{A[j,k]}_{C(j,k)} \cdot E_{j,k}.$$

Definition 2.3.4 *The* **dimension** *of a (complex) vector space is the number of elements in a basis of the vector space.*

- In general, $\mathbb{R}^n$ has dimension n as a real vector space.
- $\mathbb{C}^n$ has dimension n as a complex vector space.
- $\mathbb{C}^{m\times n}$: the dimension is mn as a complex vector space.
- The dimension of $\mathbb{V} \times \mathbb{V}'$ is the dimension of $\mathbb{V}$ plus the dimension of $\mathbb{V}'$.

$\mathbb{C}^4$

Example 4: a) Find the coefficient of the vector $v = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$ with respect to the basis

$$B = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -2 \\ 4 \end{bmatrix}} \right\}.$$

b) Find the coefficient of the vector v with respect to the basis $D = \left\{ \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \end{bmatrix} \right\}$?

$$v = c_1 v_1 + c_2 v_2$$

$$\begin{bmatrix} 7 \\ -17 \end{bmatrix} = \begin{bmatrix} c_1 - 2c_2 \\ -3c_1 + 4c_2 \end{bmatrix}$$

$$\begin{cases} c_1 - 2c_2 = 7 \quad (2) \\ -3c_1 + 4c_2 = -17 \end{cases}$$

$$-c_1 = -3 \Rightarrow \boxed{c_1 = 3}$$

$$-2c_2 = 4$$

$$3 - 2c_2 = 7$$

$$\boxed{c_2 = -2}$$

1.4 Inner product and Hilbert Space

Tuesday, September 3, 2024 6:35 PM

Definition 2.4.1 *An* **inner product** *(also called a* **dot product** *or* **scalar product***) on a complex vector space* $\mathbb{V}$ *is a function* $f: \mathbb{V}_1 \times \mathbb{V}_2 \longrightarrow \mathbb{C}$

$$f(v_1, v_2) = \underline{\langle v_1, v_2 \rangle} = v_1{}^{\dagger} * v_2 = \sum_{i=1}^{n} \overline{c_i}\, c'_i$$

where

$$v_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \qquad v_2 = \begin{bmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \overline{c_1} & \overline{c_2} & \cdots & \overline{c_n} \end{bmatrix}}_{1\times n} \underbrace{\begin{bmatrix} c'_1 \\ c'_2 \\ \vdots \\ c'_n \end{bmatrix}}_{n\times 1}$$

$$= \overline{c_1} c'_1 + \overline{c_2} \cdot c'_2 + \cdots + \overline{c_n} c'_n$$

Example 1: Find $\langle v_1, v_2 \rangle$

$$v_1 = \begin{bmatrix} 2+i \\ ⬚ \\ 3-2i \end{bmatrix} \qquad v_1 = \begin{bmatrix} -3 \\ ⬚ \\ 2i \end{bmatrix}$$

$$\langle v_1, v_2 \rangle = \begin{bmatrix} 2-i & 3+2i \end{bmatrix} \begin{bmatrix} -3 \\ 2i \end{bmatrix}$$

$$= -6 + 3i + 6i - 4 = -10 + 9i$$

The inner product function $f: \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{C}$ satisfies the following conditions :
For all $v, v_1, v_2, and\ v_3 \in \mathbb{V}$ and $c \in \mathbb{C}$

(i) *Nondegenerate:*

$\langle V, V \rangle \geq 0,$

$\langle V, V \rangle = 0$ *if and only if* $V = \mathbf{0}$

(i.e., the only time it "degenerates" is when it is 0).

$$V = \begin{bmatrix} 2+i \\ 3-i \end{bmatrix} \Rightarrow \langle V, V \rangle = \begin{bmatrix} 2-i & 3+i \end{bmatrix} \begin{bmatrix} 2+i \\ 3-i \end{bmatrix}$$

$$= 5 + 10 = \underline{\underline{15}}$$

(ii) *Respects addition:*

$\langle V_1 + V_2, V_3 \rangle = \langle V_1, V_3 \rangle + \langle V_2, V_3 \rangle,$

$\langle V_1, V_2 + V_3 \rangle = \langle V_1, V_2 \rangle + \langle V_1, V_3 \rangle.$

$$\langle v_1 + v_2, v_3 \rangle = (v_1 + v_2)^{\dagger} * v_3$$

$$= (v_1^{\dagger} + v_2^{\dagger}) * v_3$$

$$= v_1^{\dagger} * v_3 + v_2^{\dagger} * v_3$$

$$= \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$$

(iii) *Respects scalar multiplication:*

$\langle c \cdot V_1, V_2 \rangle = \overline{c} \times \langle V_1, V_2 \rangle,$

$$\langle c \cdot v_1, v_2 \rangle = (c \cdot v_1)^{\dagger} \cdot v_2 = \overline{c}\, (v_1^{\dagger} \cdot v_2)$$

$$= \overline{c} \times \langle v_1, v_2 \rangle$$

$\langle V_1, c \cdot V_2 \rangle = c \times \langle V_1, V_2 \rangle.$

$$\langle V_1, c V_2 \rangle$$

$$= V_1^{\dagger} * c * V_2$$

$$= c \times (V_1^{\dagger} * V_2)$$

$$= c \times \langle V_1, V_2 \rangle$$

(iv) *Skew symmetric:*

$\langle V_1, V_2 \rangle = \overline{\langle V_2, V_1 \rangle}.$

$$\overline{(V_2^{\dagger}\, V_1)} = \overline{V_2^{\dagger}} \cdot \overline{V_1} = (V_2^{T} \times \overline{V_1})^{T} \qquad (\overline{V_2^{\dagger}}) = (V_2^{T})$$

$$(\overline{V_1})^{T} \cdot (V_2^{T})^{T}$$

$$= V_1^{\dagger}\, V_2 = \langle V_1, V_2 \rangle$$

Definition (**Trace**): the trace of a square matrix C is given by the sum of the diagonal elements

$$Trace(C) = \sum_{i=0}^{n} c[i, i]$$

Example 2: Find the trace of the matrix

$$A = \begin{bmatrix} 3-i & 2 & i \\ -i & 0 & 4 \\ 7 & 2-i & i \end{bmatrix} \Longrightarrow Trace(A) = 3 - i + 0 + i = \underline{\underline{3}}$$

Definition : The inner product given for matrices $A, B \in \mathbb{C}^{m\times n}$

$$\langle A, B\rangle = Trace(A^{\dagger} * B)$$

Example 3: Find the inner product of two matrices

$$A = \begin{bmatrix} i & 2-i \\ 3 & -i \end{bmatrix} \qquad B = \begin{bmatrix} 1+i & 4 \\ 5-i & 2+3i \end{bmatrix}$$

$$A^{\dagger} * B = \begin{bmatrix} -i & 3 \\ 2+i & i \end{bmatrix} \begin{bmatrix} 1+i & 4 \\ 5-i & 2+3i \end{bmatrix} = \begin{bmatrix} \boxed{16-4i} & \\ & \boxed{5+6i} \end{bmatrix}$$

1x1 1x2 2x1 2x2

$$\langle A, B\rangle = 16 - 4i + 5 + 6i = 21 + 2i$$

■ $\mathbb{R}^{n\times n}$ has an inner product given for matrices $A, B \in \mathbb{R}^{n\times n}$ as

$$\langle A, B\rangle = Trace(A^T \star B),$$

Definition 2.4.3 *For every complex inner product space $\mathbb{V}$, $\langle -, -\rangle$, we can define a* **norm** *or* **length** *which is a function*

$V \in \mathbb{V}$ $\quad |V|$

$$|\ \ | : \mathbb{V} \longrightarrow \mathbb{R}$$

$$\textit{defined as } |V| = \sqrt{\langle V, V\rangle}.$$

Exercise 2.4.5 Calculate the norm of $V = [4 + 3i, 6 - 4i, 12 - 7i, 13i]^T$. $= \begin{bmatrix} 4+3i \\ 6-4i \\ 12-7i \\ 13i \end{bmatrix}$

$$|V| = \sqrt{\langle V, V\rangle} = \sqrt{4^2+3^2+6^2+4^2+12^2+7^2+13^2} = 20\ 952$$

$$\begin{bmatrix} 4-3i & 6+4i & 12+7i & -13i \end{bmatrix} \begin{bmatrix} 4+3i \\ 6-4i \\ 12-7i \\ 13i \end{bmatrix}$$

From the properties of an inner product space, it follows that a norm has the following properties for all $V, W \in \mathbb{V}$ and $c \in \mathbb{C}$:

$\sqrt{\langle V, V\rangle} \geq 0$

(i) Norm is nondegenerate: $|V| > 0$ if $V \neq \mathbf{0}$ and $|\mathbf{0}| = 0$.
(ii) Norm satisfies the **triangle inequality**: $|V + W| \leq |V| + |W|$.
(iii) Norm respects scalar multiplication: $|c \cdot V| = |c| \times |V|$. $= \sqrt{c * \bar{c}}$ $\quad \sqrt{\langle V, V\rangle}$

(ii) Norm satisfies the **triangle inequality**: $|V + W| \le |V| + |W|$.
(iii) Norm respects scalar multiplication: $|c \cdot V| = |c| \times |V|$. $= \sqrt{c \times \bar{c}}$ $\quad \sqrt{\langle V, V\rangle}$

Example 6: Let $A = \begin{bmatrix} 2i & -1 \\ 3+i & 5 \end{bmatrix}$, find the norm of A

$$A^{\dagger} * A = \begin{bmatrix} -2i & 3-i \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ 3+i & 5 \end{bmatrix} = \begin{bmatrix} 14 & \\ & 26 \end{bmatrix}$$

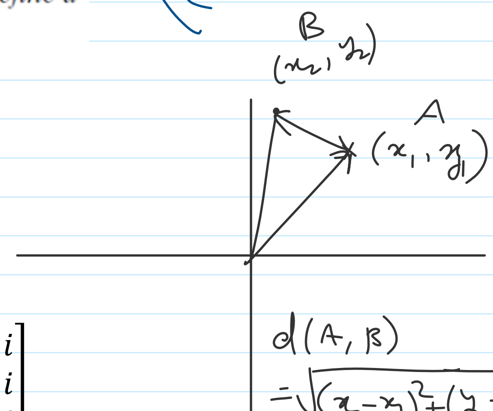
$$|A| = \sqrt{\langle A, A\rangle} = \sqrt{\mathrm{Trace}(A^{\dagger} * A)} = \sqrt{40} = 2\sqrt{10}$$

Definition 2.4.4 *For every complex inner product space* $(\mathbb{V}, \langle\ ,\ \rangle)$, *we can define a* **distance function**

$$d(\ ,\ ) : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R},$$

where

$$d(V_1, V_2) = |V_1 - V_2| = \sqrt{\langle V_1 - V_2, V_1 - V_2\rangle}.$$



$d(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$2i - 4 - i$

Example 2.4.5: Find the distance between $v_1 = \begin{bmatrix} 2i \\ 1-i \\ 3 \end{bmatrix}$ $-$ *and* $v_2 = \begin{bmatrix} 4+i \\ 7+i \\ 2-i \end{bmatrix}$

$1 - i - 7 - i$

$$d(v_1, v_2) = |v_1 - v_2|$$
$$= \sqrt{\langle v_1 - v_2, v_1 - v_2\rangle}$$
$$= \sqrt{4^2 + 1^2 + 6^2 + 2^2 + 1^2 + 1^2} = \sqrt{59}$$

$$v_1 - v_2 = \begin{bmatrix} -4+i \\ -6-2i \\ 1+i \end{bmatrix}$$

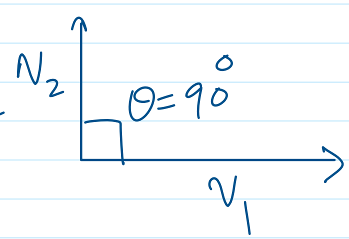
Note: The distance function satisfied the following properties, for $V, U, W \in \mathbb{V}$

(i) Distance is nondegenerate: $d(V, W) > 0$ if $V \neq W$ and $d(V, V) = 0$.
(ii) Distance satisfies the **triangle inequality**: $d(U, V) \le d(U, W) + d(W, V)$.
(iii) Distance is symmetric: $d(V, W) = d(W, V)$.

In Real Number space $\langle V, V'\rangle = |V||V'| \cos\theta$, where θ is the angle between V and V'.

Definition 2.4.5 *Two vectors* V_1 *and* V_2 *in an inner product space* $\mathbb{V}$ *are* **orthogonal** *if* $\langle V_1, V_2\rangle = 0$.

$$\langle V_1, V_2\rangle = |V_1||V_2| \overbrace{\cos(90^\circ)}^{=0} = 0$$



Example 7: Determine if each pair of states is orthogonal or not.

a) $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *and* $v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\quad$ b) $v_1 = \begin{bmatrix} \frac{1-\sqrt{3}i}{4} \\ \frac{\sqrt{2}+i}{2} \end{bmatrix}$ *and* $v_2 = \begin{bmatrix} \frac{2+i}{2} \\ \frac{-1+\sqrt{3}i}{4} \end{bmatrix}$

ⓐ $\langle V_1, V_2 \rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = -i \neq 0$ Not orthogonal.

ⓑ

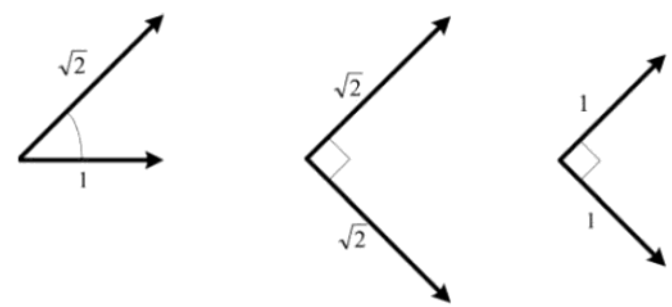
Definition 2.4.6 *A basis $\mathcal{B} = \{V_0, V_1, \ldots, V_{n-1}\}$ for an inner product space $\mathbb{V}$ is called an* **orthogonal basis** *if the vectors are pairwise orthogonal to each other, i.e., $j \neq k$ implies $\langle V_j, V_k \rangle = 0$. An orthogonal basis is called an* **orthonormal basis** *if every vector in the basis is of norm 1, i.e.,*

$$\langle V_j, V_k \rangle = \delta_{j,k} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

$\Rightarrow \langle V_2, V_2 \rangle = 1 \qquad |V_2| = \sqrt{1} = 1$

$\langle V_2, V_3 \rangle = 0$

$\delta_{j,k}$ is called the **Kronecker delta function**.



(i) Not orthogonal (ii) Orthogonal but not orthonormal (iii) Orthonormal

Example 8: Consider the three bases, determine the orthogonal and orthonormal

(i) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

(ii) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

(iii) $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$E_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad V = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$

Proposition 2.4.1 In $\mathbb{C}^n$, we also have that any V can be written as

$$V = \underbrace{\langle E_0, V \rangle}_{c_0} E_0 + \underbrace{\langle E_1, V \rangle}_{c_1} E_1 + \cdots + \underbrace{\langle E_{n-1}, V \rangle}_{c_{n-1}} E_{n-1}.$$

$\begin{bmatrix} c_0 \\ 0 \\ \vdots \end{bmatrix}$

It must be stressed that this is true for any orthonormal basis, not just the canonical one.

Definition 2.4.9 *A* **Hilbert space** *is a complex inner product space that is complete.*

Proposition 2.4.2 Every inner product on a *finite*-dimensional complex vector space is automatically complete; hence, every finite-dimensional complex vector space with an inner product is automatically a Hilbert space.

1.5 Eigenvalues and Eigenvectors

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Example 2.5.1: Let $A = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix}$ compute the product Av for each $v_1 = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(a) (b)

$$Av_1 = \begin{bmatrix} -50 \\ -40 \\ 30 \end{bmatrix} = 10 v_1$$

$$Av_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 v_2$$

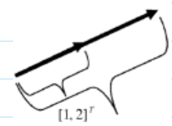
$$Av = cv$$

c → eigenvalue

v → eigenvector

Definition 2.5.1 *For a matrix A in $\mathbb{C}^{n\times n}$, if there is a number c in $\mathbb{C}$ and a vector $V \neq 0$ with $\mathbb{C}^n$ such that*

$$AV = c \cdot V,$$



then c is called an **eigenvalue** *of A and V is called an* **eigenvector** *of A associated with c. ("eigen-" is a German prefix that indicates possession.)*

If a matrix A has eigenvalue c_0 with eigenvector V_0, then for any $c \in \mathbb{C}$ we have

$$A(cV_0) = cAV_0 = cc_0V_0 = c_0(cV_0),$$

which shows that cV_0 is also an eigenvector of A with eigenvalue c_0.

Proposition 2.5.1 Every eigenvector determines a complex subvector space of the vector space. This space is known as the **eigenspace** associated with the given eigenvector.

Theorem 2.5.1: Let A be $n \times n$ matrix and suppose $\det(cI - A) = 0$ for some $c \in \mathbb{C}$. Then c is an eigenvalue of A and thus there exists a **nonzero** vector $v \in \mathbb{C}^n$ such that $Av = cv$

$$cv - Av = 0$$

$$(cI - A)v = 0 \Rightarrow \det(cI - A) = 0$$

Procedure: Finding Eigenvalues and Eigenvectors for A be $n \times n$ matrix.

1. Find the eigenvalue c of A by solving the equation $\det(cI - A) = 0$.
2. For each c, find the basic eigenvector $v \neq 0$ by finding the solution of $(cI - A)v = 0$.

Example 2.5.2: Let $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

c c

$$\det(cI - A) = 0 \qquad cI = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

$$cI = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

$$\det(cI - A) = 0$$

$$= \det\left(\begin{bmatrix} c+5 & -2 \\ 7 & c-4 \end{bmatrix}\right) = 0$$

$$(c+5)(c-4) + 14 = 0$$

$6 = 3 \times 2$

$6 \times 1 \times$

$3 - 2 = 1$

$6 - 1 = 5 \times$

$$c^2 + c - 20 + 14 = 0$$

$$c^2 + c - 6 = 0 \Rightarrow c^2 + c + \left(\tfrac{1}{2}\right)^2 = 6 + \tfrac{1}{4}$$

$$(c+3)(c-2) = 0 \qquad \left(c + \tfrac{1}{2}\right)^2 = \tfrac{25}{4}$$

$$\boxed{c = -3} \text{ or } \boxed{c = 2}. \qquad c + \tfrac{1}{2} = \pm \tfrac{5}{2}$$

$$c = \pm \tfrac{5}{2} - \tfrac{1}{2} \Rightarrow c_1 = 2, \; c_2 = -3$$

If $c_1 = -3$:

$$(c_1 I - A) V_1 = 0$$

$$\begin{bmatrix} 2 & -2 \\ 7 & -7 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{matrix} 2v_1 - 2v_2 = 0 \Rightarrow v_1 = v_2 \\ 7v_1 - 7v_2 = 0 \end{matrix}$$

$$\therefore V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1+i \\ 1+i \end{bmatrix}$$

If $c_2 = 2$

$$\begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = 0 \Rightarrow \begin{matrix} 7v_1' - 2v_2' = 0 \Rightarrow v_1' = \frac{2}{7} v_2' \\ 7v_1' - 2v_2' = 0 \end{matrix}$$

$$\text{eigenvector } V' = \begin{bmatrix} \frac{2}{7} v_2' \\ v_2' \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \qquad v_2' = 7$$

2.6 Hermitian and Unitary Matrices

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A matrix $A \in \mathbb{R}^{n\times n}$ is called **symmetric** if $A^T = A$. In other words, $A[j,k] = A[k,j]$. Let us generalize this notion from the real numbers to the complex numbers.

Definition 2.6.1 *An n-by-n matrix A is called* **Hermitian** *if* $A^\dagger = A$. *In other words,* $A[j,k] = \overline{A[k,j]}$.

Definition 2.6.2 *If A is a Hermitian matrix then the operator that it represents is called* **self-adjoint**.

Example 2.6.1: Determine whether the following matrix is Hermitian matrix or not

$$A = \begin{bmatrix} 5 & 4+5i & 6-16i \\ 4-5i & 13 & 7 \\ 6+16i & 7 & -2.1 \end{bmatrix} = A^{\dagger}$$

Proposition 2.6.1 If A is a Hermitian n-by-n matrix, then for all $V, V' \in \mathbb{C}^n$ we have

$$\langle AV, V'\rangle = \langle V, AV'\rangle.$$

$$\langle AV, V'\rangle = (AV)^{\dagger} \cdot V' = V^{\dagger} \cdot A^{\dagger} \cdot V'$$
$$= V^{\dagger} \cdot A V' = \langle V, AV'\rangle$$

Proposition 2.6.2 If A is a Hermitian, then all eigenvalues are real.

To prove this, let A be a Hermitian matrix with an eigenvalue $c \in \mathbb{C}$ and an eigenvector V. Consider the following sequence of equalities:

Let c and V are eigenvalue and eigenvector of A

$AV = cV$.

$$\underline{\bar{c}\langle V, V\rangle} = \langle cV, V\rangle = \langle AV, V\rangle$$
$$= \langle V, AV\rangle = \langle V, cV\rangle = \underline{c\langle V, V\rangle}$$

$$(\bar{c} - c)\langle V, V\rangle = 0, \quad \langle V, V\rangle \neq 0$$

$\bar{c} - c = 0 \Rightarrow \bar{c} = c \quad \therefore c \in \mathbb{R}$

$(\bar{c} - c)\langle v, v\rangle = 0$

$\bar{c} - c = 0 \Rightarrow \bar{c} = c \therefore c \in \mathbb{R}$

Proposition 2.6.3 For a given Hermitian matrix A, distinct eigenvectors that have distinct eigenvalues are orthogonal.

Let c_1 be eigenvalue associate with v_1

c_1 & c_2 are real

c_2 " " " " v_2.

(prop 2.6.2) $c_1\langle v_1, v_2\rangle = \langle c_1 v_1, v_2\rangle = \langle A v_1, v_2\rangle$

$= \langle v_1, A v_2\rangle = c_2\langle v_1, v_2\rangle$

$(c_1 - c_2)\langle v_1, v_2\rangle = 0$

$\langle v_1, v_2\rangle = 0$

Definition 2.6.3 *A* **diagonal matrix** *is a square matrix whose only nonzero entries are on the diagonal. All entries off the diagonal are zero.*

Proposition 2.6.4 (The Spectral Theorem for Finite-Dimensional Self-Adjoint Operators.) Every self-adjoint operator A on a finite-dimensional complex vector space $\mathbb{V}$ can be represented by a diagonal matrix whose diagonal entries are the eigenvalues of A, and whose eigenvectors form an orthonormal basis for $\mathbb{V}$ (we shall call this basis an **eigenbasis**).

Note: Hermitian matrices with their eigenbasis play important role in quantum mechanics.
- For every physical observable of quantum system there is a corresponding Hermitian matrix.
- Measurements of that observable always lead to a state that is represented by one of the eigenvectors of the associated Hermitian matrix.

$A v = v' = c v$

Definition 2.6.4 *An n-by-n matrix U is* **unitary** *if*

$$U \star U^{\dagger} = U^{\dagger} \star U = I_n.$$

It is important to realize that not all invertible matrices are unitary.

Example 2.6.2: Determine whether the following matrices are unitary or not

$$A = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1+i}{2} & \frac{i}{\sqrt{3}} & \frac{3+i}{2\sqrt{15}} \\ \frac{-1}{2} & \frac{1}{\sqrt{3}} & \frac{4+3i}{2\sqrt{15}} \\ 1 & -i & i \end{bmatrix}$$

$\cos^2\theta + \sin^2\theta = 1$

$$A = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{\ }{2} & \frac{\ }{\sqrt{3}} & \frac{\ }{2\sqrt{15}} \\ \frac{1}{2} & \frac{-i}{\sqrt{3}} & \frac{i}{2\sqrt{15}} \end{bmatrix}$$

$(\cos^2\theta + \sin^2\theta = 1)$

$$AA^{\dagger} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Proposition 2.6.5 Unitary matrices preserve inner products, i.e., if U is unitary, then for any $V, V' \in \mathbb{C}^n$, we have $\langle UV, UV' \rangle = \langle V, V' \rangle$.

Also, $|UV| = \sqrt{\langle UV, UV \rangle} = \sqrt{\langle V, V \rangle} = |V|.$

$$\langle UV, UV' \rangle = V^{\dagger} \underline{U^{\dagger} U} V' = \langle V, V' \rangle$$

2.7 Tensor Product of Vector Spaces

Wednesday, September 11, 2024 7:10 PM

Given two vector spaces $\mathbb{V}$ and $\mathbb{V}'$of dimensions n and m, respectively. The **tensor product** of two vector spaces, and denote it $\mathbb{V} \otimes \mathbb{V}'$.
Let $\mathrm{V} \in \mathbb{V}$ and $\mathrm{V}' \in \mathbb{V}'$, and $\mathrm{V} = c_0 E_0 + c_1 E_1 + \cdots + c_{n-1} E_{n-1}$, $\mathrm{V} = c'_0 E'_0 + c'_1 E'_1 + \cdots + c'_{m-1} E'_{m-1}$

$$V \bigotimes V' = \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \left(c_i \times c'_j\right) \left(E_i \bigotimes E_j \right)$$

Example 2.7.1

$$V_1 = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \quad V_2 = \begin{bmatrix} i \\ 1-3i \\ 0 \end{bmatrix}$$

$$V_1 \otimes V_2 = \begin{bmatrix} 1+i \; V_2 \\ 2 \; V_2 \end{bmatrix} = \begin{bmatrix} -1+i \\ 4-2i \\ 0 \\ 2i \\ 2-6i \\ 0 \end{bmatrix}$$

$|01\rangle \quad 0 | \rightarrow 1$

$|0\rangle \otimes |1\rangle$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle$$

The **tensor product** of two matrices:

Formally, the tensor product of matrices is a function

$$\otimes : \mathbb{C}^{m \times m'} \times \mathbb{C}^{n \times n'} \longrightarrow \mathbb{C}^{mn \times m'n'}$$

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix}.$$

$$A \otimes B = \begin{bmatrix} a_{0,0} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} & a_{0,1} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} \\ a_{1,0} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} & a_{1,1} \cdot \begin{bmatrix} b_{0,0} & b_{0,1} & b_{0,2} \\ b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} a_{0,0} \times b_{0,0} & a_{0,0} \times b_{0,1} & a_{0,0} \times b_{0,2} & a_{0,1} \times b_{0,0} & a_{0,1} \times b_{0,1} & a_{0,1} \times b_{0,2} \\ a_{0,0} \times b_{1,0} & a_{0,0} \times b_{1,1} & a_{0,0} \times b_{1,2} & a_{0,1} \times b_{1,0} & a_{0,1} \times b_{1,1} & a_{0,1} \times b_{1,2} \\ a_{0,0} \times b_{2,0} & a_{0,0} \times b_{2,1} & a_{0,0} \times b_{2,2} & a_{0,1} \times b_{2,0} & a_{0,1} \times b_{2,1} & a_{0,1} \times b_{2,2} \\ a_{1,0} \times b_{0,0} & a_{1,0} \times b_{0,1} & a_{1,0} \times b_{0,2} & a_{1,1} \times b_{0,0} & a_{1,1} \times b_{0,1} & a_{1,1} \times b_{0,2} \\ a_{1,0} \times b_{1,0} & a_{1,0} \times b_{1,1} & a_{1,0} \times b_{1,2} & a_{1,1} \times b_{1,0} & a_{1,1} \times b_{1,1} & a_{1,1} \times b_{1,2} \\ a_{1,0} \times b_{2,0} & a_{1,0} \times b_{2,1} & a_{1,0} \times b_{2,2} & a_{1,1} \times b_{2,0} & a_{1,1} \times b_{2,1} & a_{1,1} \times b_{2,2} \end{bmatrix}.$$

Exercise 2.7.3 Calculate

$$\begin{bmatrix} 3+2i & 5-i & 2i \\ 0 & 12 & 6-3i \\ 2 & 4+4i & 9+3i \end{bmatrix} \otimes \begin{bmatrix} 1 & 3+4i & 5-7i \\ 10+2i & 6 & 2+5i \\ 0 & 1 & 2+9i \end{bmatrix}.$$

The tensor product satisfies several useful properties:

1. Distributive: $V \otimes (U + W) = (V \otimes U) + (V \otimes W)$,
   similarly, $(V + U) \otimes W = (V \otimes W) + (U \otimes W)$.
2. Associative: $V \otimes (U \otimes W) = (V \otimes U) \otimes W$.
3. Not Commutative: In general $V \otimes U \neq U \otimes V$