1.2 Complex Vector Space

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Representation of Vectors over Complex Numbers

$V \in \mathbb{C}^n \quad V = (v_1, v_2, v_3, \ldots v_n)$

$\mathbb{V} = \mathbb{C}^2$

$u = (1+i, 3) \quad v = (1-4i, 2+i)$

$u + v = (2-3i, 5+i)$

Definition 2.2.1: A vector space $\mathbb{V} = \mathbb{C}^n$ over $\mathbb{C}$ is a set $\mathbb{V}$ with two operations:

- Vector addition: for every $v, u \in \mathbb{V}$, there is an element $v + u \in \mathbb{V}$
- Scalar multiplication: for every $c \in \mathbb{C}$ and $v \in \mathbb{V}$, there is an element $c \cdot v \in \mathbb{V}$ such that the following axioms hold: $c = (1+i) \quad v = (1-4i, 2+i) \quad cv \in \mathbb{C}^2$

I. $v + u = u + v$ for every $v, u \in \mathbb{V}$ (commutativity of addition)

II. $(v + u) + w = v + (u + w)$ for every $v, u, w \in \mathbb{V}$ (associativity of addition) $\quad v = (1-4i, 2+i) + (0,0) = v$

III. There is an element $\mathbf{0} \in \mathbb{V}$ such that $v + \mathbf{0} = v$ for every $v \in \mathbb{V}$ ($\mathbf{0}$ is vector additive identity)

IV. For every $v \in \mathbb{V}$ there is $-v \in \mathbb{V}$ such that $v + (-v) = \mathbf{0}$ (vector additive inverse) $\Rightarrow v + (-v) = 0$

V. $1 \cdot v = v$ for every $v \in \mathbb{V}$ (where 1 is multiplicative identity of $\mathbb{C}$) $\quad 1 \cdot v = v$

VI. $c_1 \cdot (c_2 \cdot v) = (c_1 \cdot c_2) \cdot v$ for every $c_1, c_2 \in \mathbb{C}$ and $v \in \mathbb{V}$ (associativity of scalar multiplication

VII. $c_1 \cdot (u + v) = c_1 \cdot u + c_1 \cdot v$ for every $c_1 \in \mathbb{C}$ and $v, u \in \mathbb{V}$ (distributivity)

VIII. $(c_1 + c_2) \cdot v = c_1 \cdot v + c_2 \cdot v$ for every $c_1, c_2 \in \mathbb{C}$ and $v \in \mathbb{V}$ (distributivity)

no of rows, no. of

Note: $\mathbb{C}^{m \times n}$, the set of all m-by-n matrices with complex entries, $\quad$ e.g. $A \in \mathbb{C}^{2 \times 3}$

$$A = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & & c_{mn} \end{bmatrix}, \quad A = \begin{bmatrix} 1+i & 2 & 0 \\ \frac{1}{2}+2i & -\frac{3}{4}-\frac{5}{2}i & 5 \end{bmatrix}$$

Exercise 2.2.3 Let $c_1 = 2i$, $c_2 = 1 + 2i$, and $A = \begin{bmatrix} 1-i & 3 \\ 2+2i & 4+i \end{bmatrix}$. Verify Properties (vi) and (viii) in showing $\mathbb{C}^{2\times 2}$ is a complex vector space.

Type equation here.

$(c_1 \, c_2) A = c_1 (c_2 A)$

$1+2i$

$c_2 A = \begin{bmatrix} 3+i & 3+6i \\ -2+6i & 2+9i \end{bmatrix}$

$2+2i$
$1+2i$

$\times \quad 1-1i$
$1+2i$

$(1+2) + i$

Definition 2.2.2: Let A $\in \mathbb{C}^{m\times n}$, the **transpose** of A is denoted by A^T and is defined as $A^T[i,j] = A[j,i]$

Definition 2.2.3: Let A $\in \mathbb{C}^{m\times n}$, the **Conjugate** of A is denoted by $\overline{A}$ and is defined as $\overline{A}[i,j] = \overline{A[i,j]}$

Definition 2.2.4: Let A $\in \mathbb{C}^{m\times n}$, the **dagger or adjoint** of A is denoted by $A^\dagger$ and is defined as $A^\dagger = \overline{A[j,i]}$

Example 2.4: Find the transpose, conjugate, and adjoint of

$A_{m\times n} = A[i,j] \quad i = 1 \to m, \; j = 1 \to n$

$$A = \begin{bmatrix} 6-3i & 2+12i & -19i \\ 0 & 5+2.1i & 17 \\ 1 & 2+5i & 3-4.5i \end{bmatrix}.$$

1) $A^T_{n\times m} = A[j,i]$

2) $\overline{A} = \overline{A[i,j]}$

$$A^T = \begin{bmatrix} 6-3i & 0 & 1 \\ 2+12i & 5+2.1i & 2+5i \end{bmatrix}$$

$$A^T = \begin{bmatrix} 6-3i & 0 & 1 \\ 2+12i & 5+2.1i & 2+5i \\ -19i & 17 & 3-4.5i \end{bmatrix}$$

2) $\overline{A} = \overline{A[i,j]}$

③ $A^\dagger = \overline{(A^T)} = \overline{A[j,i]}$

$$\overline{A} = \begin{bmatrix} 6+3i & 2-12i & +19i \\ 0 & 5-2.1i & 17 \\ 1 & 2-5i & 3+4.5i \end{bmatrix}$$

$$A^\dagger = \overline{(A^T)} = \begin{bmatrix} 6+3i & 0 & 1 \\ 2-12i & 5-2.1i & 2-5i \\ +19i & 17 & 3+4.5i \end{bmatrix}$$

These operations satisfy the following properties for all $c \in \mathbb{C}$ and for all A, $B \in \mathbb{C}^{m\times n}$:

(i) Transpose is idempotent: $(A^T)^T = A$.
(ii) Transpose respects addition: $(A+B)^T = A^T + B^T$.
(iii) Transpose respects scalar multiplication: $(c \cdot A)^T = c \cdot A^T$.

(iv) Conjugate is idempotent: $\overline{\overline{A}} = A$.
(v) Conjugate respects addition: $\overline{A+B} = \overline{A} + \overline{B}$.
(vi) Conjugate respects scalar multiplication: $\overline{c \cdot A} = \overline{c} \cdot \overline{A}$.
(vii) Adjoint is idempotent: $(A^\dagger)^\dagger = A$.
(viii) Adjoint respects addition: $(A+B)^\dagger = A^\dagger + B^\dagger$.
(ix) Adjoint relates to scalar multiplication: $(c \cdot A)^\dagger = \overline{c} \cdot A^\dagger$.

$$(\overline{C \cdot A})^T = (\overline{C\, A[i,j]})^T$$
$$= (\overline{C}\ \overline{A[i,j]})^T$$
$$= \overline{C}\ \overline{A[j,i]}$$
$$= \overline{C}\ A^\dagger$$

$$(C \cdot A)^\dagger = (\overline{C \cdot A})^T = (\overline{C} \cdot \overline{A})^T = \overline{C} \cdot A^\dagger$$

Matrix Multiplication: $\mathbb{C}^{m\times n} \times \mathbb{C}^{n\times p} \longrightarrow \mathbb{C}^{m\times p}$.

$$A_{m\times n} \times B_{n\times p} = C_{m\times p}$$

Formally, given A in $\mathbb{C}^{m\times n}$ and B in $\mathbb{C}^{n\times p}$, we construct $A \star B$ in $\mathbb{C}^{m\times p}$ as

$$C[j,k] = (A \star B)[j,k] = \sum_{h=0}^{n-1} (A[j,h] \times B[h,k]).$$

j row of A

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 4 & 5 & 7 \end{bmatrix}_{2\times 3} \qquad B = {}_{3\times 1}\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$A \times B[1,1]$
$= A[1,1]B[1,1] + A[1,2]B[2,1] + A[1,3]B[3,1]$

$$= \begin{bmatrix} 2\times 0 + 3\times -1 + 6\times 3 \\ 4\times 0 + 5\times -1 + 7\times 3 \end{bmatrix}_{2\times 1} = \begin{bmatrix} 15 \\ 16 \end{bmatrix}$$

For every n, there is a special n-by-n matrix called the **identity matrix**,

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \cdot I = A$$

Matrices multiplication properties: For all $A, B, C \in \mathbb{C}^{n\times n}$,

(i) Matrix multiplication is associative: $(A \star B) \star C = A \star (B \star C)$.
(ii) Matrix multiplication has I_n as a unit: $I_n \star A = A = A \star I_n$.
(iii) Matrix multiplication distributes over addition:

$$A \star (B + C) = (A \star B) + (A \star C),$$

$$(B + C) \star A = (B \star A) + (C \star A).$$

(iv) Matrix multiplication respects scalar multiplication:

$$c \cdot (A \star B) = (c \cdot A) \star B = A \star (c \cdot B).$$

(v) Matrix multiplication relates to the transpose:

$$(A \star B)^T = B^T \star A^T.$$

$$\left(A_{2\times 3} * B_{3\times 4}\right)^T = \left(D_{2\times 4}\right)^T = C_{4\times 2}$$

$$A^T_{3\times 2} \quad B^T_{4\times 3}$$

$$B^T_{4\times 3} * A^T_{3\times 2} = C_{4\times 2}$$

(vi) Matrix multiplication respects the conjugate:

$$\overline{A \star B} = \overline{A} \star \overline{B}.$$

(vii) Matrix multiplication relates to the adjoint:

$$(A \star B)^\dagger = B^\dagger \star A^\dagger.$$

$$\left(\overline{(A * B)}\right)^T = \left(\overline{A} * \overline{B}\right)^T = \overline{B}^T * \overline{A}^T = B^\dagger * A^\dagger$$

Exercise 2.5: Given A amd B, show that $(A * B)^\dagger = B^\dagger * A^\dagger$

$$A = \begin{bmatrix} 3+2i & 0 & 5-6i \\ 1 & 4+2i & i \\ 4-i & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 2-i & 6-4i \\ 0 & 4+5i & 2 \\ 7-4i & 2+7i & 0 \end{bmatrix}.$$

$$A^\dagger = \begin{bmatrix} 3-2i & 1 & 4+i \\ 0 & 4-2i & 0 \\ 5+6i & -i & 4 \end{bmatrix}$$

Definition 2.2.4 *Given two complex vector spaces $\mathbb{V}$ and $\mathbb{V}'$, we say that $\mathbb{V}$ is a* **complex subspace** *of $\mathbb{V}'$ if $\mathbb{V}$ is a subset of $\mathbb{V}'$ and the operations of $\mathbb{V}$ are restrictions of operations of $\mathbb{V}'$.*

$S\{0, \ldots, 9\}$

$S = \{1, 2, 3\}$

$\mathbb{V} = (\mathbb{V}, +, *)$ subspace

$\mathbb{V}' = (\mathbb{V}', +, *)$

Equivalently, $\mathbb{V}$ is a complex subspace of $\mathbb{V}'$ if $\mathbb{V}$ is a subset of the set $\mathbb{V}'$ and

(i) $\mathbb{V}$ is closed under addition: For all V_1 and V_2 in $\mathbb{V}$, $V_1 + V_2 \in \mathbb{V}$.
(ii) $\mathbb{V}$ is closed under scalar multiplication: For all $c \in \mathbb{C}$ and $V \in \mathbb{V}$, $c \cdot V \in \mathbb{V}$.

Example 2.2.7: Consider the set of all vectors of $\mathbb{C}^9$ with the second, fifth, and eighth position elements being 0:

2 5 8

$V \in \mathbb{C}^9 \quad V = (v_1, 0, v_3, v_4, 0, v_6, v_7, 0, v_9) \quad V \subseteq V$

$x^0 \quad x^1 \quad x^2 \quad x^3 \quad x$

$\mathbb{C}^9 \quad v_i \in \mathbb{C}$

$p(x) = v_1 + v_3 x^2 + v_4 x^3 + v_6 x^5 + v_7 x^6 + v_9 x^8$

Definition 2.2.5: The set of polynomials of degree n or less in one variable with coefficients in $\mathbb{C}$ is form a complex vector space.

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n.$$

Definition 2.2.5 *Let $\mathbb{V}$ and $\mathbb{V}'$ be two complex vector spaces. A* **linear map** *from $\mathbb{V}$ to $\mathbb{V}'$ is a function $f : \mathbb{V} \longrightarrow \mathbb{V}'$ such that for all V, V_1, $V_2 \in \mathbb{V}$, and $c \in \mathbb{C}$,*

$V_1, V_2 \in V$

(i) *f respects the addition: $f(V_1 + V_2) = f(V_1) + f(V_2)$,*
(ii) *f respects the scalar multiplication: $f(c \cdot V) = c \cdot f(V)$.*

Definition 2.2.6 *Two complex vector spaces $\mathbb{V}$ and $\mathbb{V}'$ are* **isomorphic** *if there is a one-to-one onto linear map $f : \mathbb{V} \longrightarrow \mathbb{V}'$. Such a map is called an* **isomorphism**.

Example 2.2.8: Show that the map $f: \mathbb{C} \longrightarrow \mathbb{R}^{2\times 2}$ define below is an isomorphic from $\mathbb{C}$ to $\mathbb{R}^{2\times 2}$

$$f(x+iy) = \begin{bmatrix} x & y \\ -y & x \end{bmatrix}$$

$V_1 = x_1 + i y_1$
$V_2 = x_2 + i y_2$

$$f(V_1 + V_2) = f((x_1+x_2) + i(y_1+y_2))$$
$$= \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ -y_1-y_2 & x_1+x_2 \end{bmatrix} \stackrel{?}{=} f(V_1) + f(V_2)$$