2.1 Permutations and Combinations

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Permutations

Example 2.1.1: (a) In how many ways can we select three students from a group of five students to stand in line for a picture? (b) In how many ways can we arrange all five of these students in a line for a picture

(a) $\boxed{5} \times \boxed{4} \times \boxed{3} = 60$ ways (b) $\boxed{5}\ \boxed{4}\ \boxed{3}\ \boxed{2}\ \boxed{1} = 120$

Example 2.1.2: How many different 4-digit code can be generated if that no digit is repeated?

$\frac{10! = 10\times 9 \times \cdots \times 2 \times 1}{6! = (10-4)!}$ $\boxed{10} \times \boxed{9} \times \boxed{8} \times \boxed{7} =$

Definition 2.1.1: If n is a positive integer and r is an integer with $1 \le r \le n$, then there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

Example 2.1.3: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 =$

Example 2.1.4: How many permutations of the letters ABCDEFGH contain the string ABC ?

1 2 3 4 5 6

$6! = 720$

$0! = 1$

$1! = 1$

Combinations

Example 2.1.5: How many different committees of three students can be formed from a group of four students?

A, B, C, D

$P(3,4) = \frac{4!}{1!} = 4! = 24$

$\boxed{A\ B\ C}$ ABD ACD BCD

A C B

$6 \leftarrow$

B A C

B C A

C A B

C B A

$$C(4,3) = \frac{P(4,3)}{6 = 3!} = 4$$

Definition 2.1.2: The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals

$$C(n,r) = \frac{n!}{r!\,(n-r)!} = \frac{P(n,r)}{r!}$$

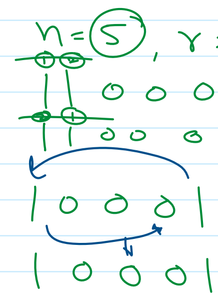
Corollary 2.1.1: Let n and r be nonnegative integers with $r \le n$. Then $C(n,r) = C(n,n-r)$.

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)!\,(n-\underbrace{(n-r)}_{n-n+r})!} = C(n,r)$$

Example 2.1.6: How many bit strings of length n contain exactly r 1s?



$$C(n,r) =$$

$n = 5,\ r = 2$

$$C(5,2) = \frac{5!}{2!\,3!} = \frac{5\times 4\times 3!}{2\times 1\times 3!} = 10.$$

Example 2.1.7: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department.

a) How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

b) How many ways are there to select a $\boxed{4}$-committee members to develop a discrete mathematics course at a school if the committee is to consist at least one faculty member from each department?

(a) $$C(9,3)\times C(11,4) = \frac{9!}{3!\,6!}\times\frac{11!}{4!\,7!}$$
$$= \frac{9\times 8\times 7}{3\times 2\times 1}\cdot\frac{11\times 10\times 9\times 8}{4\times 3\times 2\times 1} = 84\times 330$$
$$= 27720.$$

(b) $$C(9,1)\times C(11,3) + C(9,2)\times C(11,2) + C(9,3)\times C(11,1)$$
$$= 4389$$

2.2 Introduction to Probability

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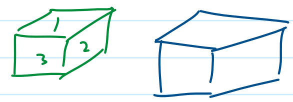
Sample space S: The sample space of the experiment is the set of possible outcomes.

Example 2.2.1: Two rolls of die, what the sample space

$$\boxed{6} \times \boxed{6} = 36$$



Event E: a subset of the sample space

Definition 2.2.1: If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the probability of E is

$$P(E) = \frac{|E|}{|S|}$$

Probability Axioms: $0 \le P(E) \le 1$

1. Nonnegative $P(E) \ge 0$.
2. Normalization $P(S) = 1$
3. Additivity: If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$

Example 2.2.2: An urn contains four blue balls and five red balls. What is the probability that a ball chosen at random from the urn is blue? $|B| = 4$, $|R| = 5$ $|S| = |B| + |R|$

$$P(B) = \frac{|B|}{|S|} = \frac{4}{9}$$

Example 2.2.3: What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

$$|S| = \underline{\overset{\boxed{6} \times 6}{36}} \qquad \boxed{6} \times \boxed{6}$$

$$E = Sum(7) = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

$$P(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

3 4 4

Example 2.2.4: From a committee of three males and four females, a subcommittee of four is to be randomly selected. Find the probability that it consists of two males and two females.

$$|S| = C(7,4) = \frac{7!}{4!\,3!} = \frac{7 \times \cancel{6} \times 5}{\cancel{3 \times 2 \times 1}} = 35$$

$$|E| = C(3,2) \times C(4,2) = 3 \times 6 = 18$$

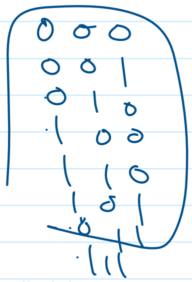
$$P(E) = \frac{18}{35} \approx 0.514$$

Probabilities of Complements and Unions of Events

Definition 2.2.2: Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$P\left(\overline{E}\right) = 1 - P(E)$$

Example 2.2.5: A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?



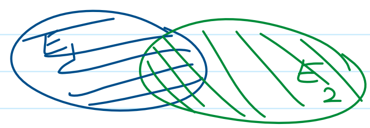
$|S| = 2^{10}$ $\quad$ E: at least one bit is zero.

$P(E) = 1 - P(\bar{E})$ $\quad$ $\bar{E}$: all bits are 1's

$= 1 - \frac{1}{2^{10}} = \frac{1023}{1024} = 0.999$

Definition 2.2.3: Let E_1 and E_2 be events in the sample space S. Then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2).$$



Example 2.2.6: What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

$S = \{1, 2, \ldots, 100\}$

$E_1 = \{\text{Integer divisible by } 2\}$ $\quad |E_1| = 50$.

$E_2 = \{\text{Integer divisible by } 5\} \Rightarrow |E_2| = 20$

$E_1 \cap E_2 = \{\text{Integer divisible by both } 2 \,\&\, 5\} \Rightarrow |E_1 \cap E_2| = 10$

$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{60}{100} = 0.60$

Remark: Let S be the sample space of an experiment with a finite or countable number of outcomes $s \in S$

1. $0 \leq P(E) \leq 1$, for every $E \in S$.
2. $\sum_{E \in S} P(E) = 1$

Example 2.2.7: (a) What probabilities should we assign to the outcomes H (heads) and T (tails) when a fair coin is flipped? (b) What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up **twice** as often as tails?

(a) $S = \{H, T\}$ $\quad P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$

(b) $P(H) = 2P(T) \Rightarrow P(H) + P(T) = 1$

$2P(T) + P(T) = 1$

$\therefore P(H) = \frac{2}{3}$ $\quad P(T) = \frac{1}{3}$

Example 2.2.8: Suppose that a die is biased (or loaded) so that 3 appears twice as often as each other number but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this die?

$\sum_{i=1}^{6} P(i) = 1$

let $p = p(j)$, $j = 1, 2, 4, 5, 6$ $\quad P(3) = 2p$.

$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

$$p + p + 2p + p + p + p = 1$$

$$\therefore p = \frac{1}{7}$$

E: odd no

$$P(E) = P(1) + P(3) + P(5)$$

$$P(1 \text{ or } 3 \text{ or } 5) = \frac{1}{7} + \frac{2}{7} + \frac{1}{7} = \frac{4}{7}$$

Theorem 2.2.1: If E_1 , E_2, . . . is a sequence of pairwise disjoint countable or finite events in a sample space S, then

$$P\left(\bigcup_i E_i\right) = \sum_i P(E_i)$$

2.3 Conditional Probability

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$F \quad E \quad P(E|F)$

Example 2.3.1: Suppose that we flip a coin three times, and all eight possibilities are equally likely. Moreover, suppose we know that the event F, that the first flip comes up tails, occurs. Given this information, what is the probability of the event E, that an odd number of tails appears?

2^3

$S = \{\underline{HHH}, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$F = \{\underline{TH}H, THT, TTH, \underline{TTT}\}$

$E = \{HHT, HTH, THH, TTT\}$

$P(E|F) = \frac{2}{4} = \frac{1}{2}$. $\quad \frac{P(E \cap F)}{P(F)}$

Definition 2.3.1: Let E and F be events with $P(F) > 0$. The conditional probability of E given F, denoted by $P(E \mid F)$, is defined as

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}$$

$P(H) = \frac{2}{3}, \quad P(T) = \frac{1}{3}$

$P(HHH) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

$P(HTH) = \frac{4}{27}$

Example 2.3.2: A bit string of length four is generated at random so that each of the 16 bit strings of length four is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0? (We assume that 0 bits and 1 bits are equally likely.)

$\{0, 1\} \to P(0) = \frac{1}{2}$

$S = \{s_0 = 0000, s_1 = 1000, \ldots, s_{16} = 1111\} \quad |S| = 16$. $\quad 1/16$. $\quad P(1) = \frac{1}{2})$

$E = \{0000, 0001, 0010, 0100, 1000, 0011, 1001, 1100\}$

$F = \{0000, 0001, 0010, 0100, 0011, 0101, 0110, 0111\}$

$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{5/16}{8/16}$ #

Example 2.3.3: What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl. (Note that BG represents a family with an older boy and a younger girl while GB represents a family with an older girl and a younger boy.)

$S = \{BB, BG, GB, GG\}$

$E = \{BB\}, \quad F = \{BB, BG, GB\}$

$P(E|F) = \frac{1}{3}$

$P(E|F) = \frac{P(E \cap F)}{P(F)}$

$\| P(E)$

<u>Independent Events</u>

Definition 2.3.2: The events E and F are independent if and only if $\boxed{P(E \cap F) = P(E)P(F)}$.

$$P(E|F) = P(E), \quad P(F|E) = P(F)$$

Example 2.3.4: Suppose E is the event that a randomly generated bit string of length four begins with a 1 and F is the event that this bit string contains an even number of 1s. Are E and F independent, if the 16 bit strings of length four are equally likely?

$$E = \{1111, 1110, 1101, 1011, 1100, 1010, 1001, 1000\}$$

$$F = \{1111, 1100, 1001, 1010, 0101, 0110, 0011, 0000\}$$

$$P(E \cap F) = \frac{4}{16} = \frac{1}{4} \qquad P(E) = \frac{8}{16} = \frac{1}{2}, \ P(F) = \frac{1}{2}$$

$\therefore$ Independent

Example 2.3.5: Are the events E, that a family with three children has children of both sexes, and F, that this family has at most one boy, independent? Assume that the eight ways a family can have three children are equally likely.

$$E = \{BBG, BGB, GBB, BGG, GBG, GGB\} \Rightarrow P(E) = \frac{6}{8} = \frac{3}{4}$$

$$F = \{GGG, BGG, GBG, GGB\} \Rightarrow P(F) = \frac{4}{8} = \frac{1}{2}$$

$$E \cap F = \{BGG, GBG, GGB\} = 3/8$$

Independent

Definition 2.3.3: The events $E_1, E_2, \ldots E_n$ are pairwise independent if and only if $P(E_i \cap E_j) = P(E_i)P(E_j)$ for all pairs of integers i and j with $1 \leq i < j \leq n$.

Bernoulli Trials and the Binomial Distribution

Each performance of an experiment with two possible outcomes is called a **Bernoulli trial**, In general, a possible outcome of a Bernoulli trial is called a success or a failure. If p is the probability of a success and q is the probability of a failure.

$$p = 1/3, \quad q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

Theorem 2.3.1: The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is

8, 2 $\quad p = \frac{8}{10}, \ q = 2/10$

$$(a+b)^n = \sum_{k=0}^{n} C(n,k) a^{n-k} b^k \qquad C(n,k)p^k q^{n-k}$$

$$P(x = \text{success}) = C(100, \ ) = p^k q^{n-k}$$

Binomial distribution is a function of k defined as $b(k; n, p) = C(n,k)p^k q^{n-k}$

Example 2.3.6: A coin is biased so that the probability of heads is 2/3. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent?

$$P(H) = \frac{2}{3}, \quad P(T) = \frac{1}{3} \quad n = 7$$

$4 \quad n-4$

$P(H)=\frac{2}{3}$, $P(T)=\frac{1}{3}$ $n=7$

$$P(k=4)=C(7,4)\,p^4 q^{n-4}$$

$$=\frac{\overset{7\times6\times5\times4!}{7!}}{\cancel{4!}\,\cancel{3!}}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^3=35\,\frac{16}{3^7}=0.2560$$

Example 2.3.7: Suppose that the probability that a 0 bit is generated is 0.9, that the probability that a 1 bit is generated is 0.1, and that bits are generated independently. What is the probability that exactly eight 0 bits are generated when 10 bits are generated?

$p=P(0)=0.9$, $q=P(1)=0.1$, $n=10$

$$P(k_0=8)=C(10,8)(0.9)^8(0.1)^2=0.1937$$

Ex: Find the probability that at least one zero when when 10 bits generated

$$P(k_0\geq 1)=1-P(k_1=10)$$

ALL are 1's

$$=1-C(10,10)(0.1)^{10}$$

$$=0.999\ldots$$

Ex: Find the probability at most three 0's

$$P(k_0=0)+P(k_0=1)+P(k_0=2)+P(k_0=3)$$

$2^{10}=1024$ $X:S\to R$

Random Variables

Definition 2.3.4: A random variable is a function from the sample space of an experiment to the set of real numbers. That is, a random variable assigns a real number to each possible outcome.

Example 2.3.8: Suppose that a coin is flipped three times. Let $X(t)$ be the random variable that equals the number of heads that appear when t is the outcome. Then $X(t)$ takes on the following values:

X	
3	–
2	
1	
0	

$\{HHH\}\to 3$

$S=\{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$f:$ $f(HHH)=3$ → 3, 2, 1, 0

$X=\{3,2,1,0\}$

$P(X=3)$

Definition 2.3.5: The distribution of a random variable X on a sample space S is the set of pairs $(r, P(X=r))$ for all $r\in X(S)$, where $P(X=r)$ is the probability that X takes the value r. (The set of pairs in this distribution is determined by the probabilities $P(X=r)$ for $r\in X(S)$.)

Ex: Find the distribution of Ex 2.3.8

Distribution

Ex: Find the distribution of Ex 2.3.8

Distribution

	X	P(X)
{HHH}	3	1/8
{HHT, HTH, THH}	2	3/8
{HTT, THT, TTH}	1	3/8
{TTT}	$\to$ 0	1/8

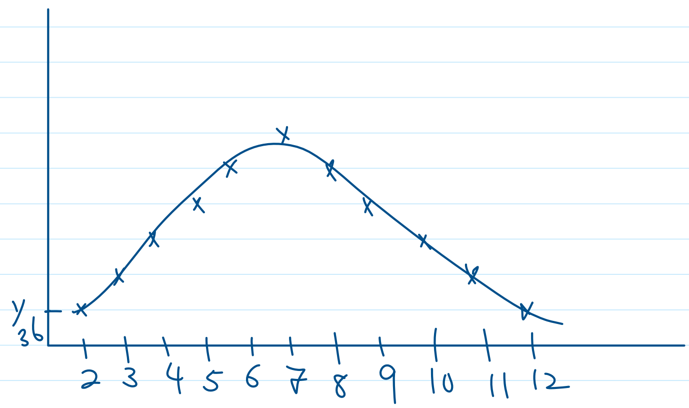
$\Sigma = \frac{8}{8} = 1$

$\sum P(X = x_i) = 1$

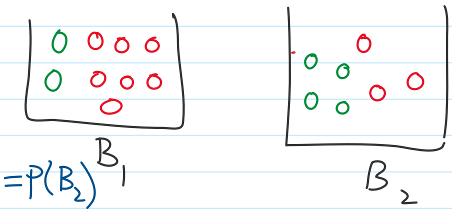
Example 2.3.6: Let X be the sum of the numbers that appear when a pair of dice is rolled. Find the distribution of X?

X=sum(a,b)

X	P(X)
2	1/36
3	2/36
4	3/36
5	4/36
6	5/36
7	6/36
8	5/36
9	4/36
10	3/36
11	2/36
12	1/36



Example 2.4.1: We have two boxes. The first contains two green balls and seven red balls; the second contains four green balls and three red balls. Bob selects a ball by first choosing one of the two boxes at random. He then selects one of the balls in this box at random. If Bob has selected a red ball, what is the probability that he selected a ball from the first box?



$$P(B_1 | R) = \frac{P(B_1 \cap R)}{P(R)}$$ ← possible outcomes

$P(B_1) = \frac{1}{2} = P(B_2)$

$P(R|B_1) = \frac{7}{9}$, $P(R|B_2) = \frac{3}{7}$

$$P(B_1|R) = \frac{P(B_1 \cap R)}{P(R \cap B_1) + P(R \cap B_2)}$$

$$= \frac{P(B_1)P(R|B_1)}{P(B_1)P(R|B_1) + P(B_2)P(R|B_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{7}{9}}{\frac{1}{2} \cdot \frac{7}{9} + \frac{1}{2}\frac{3}{7}} = \frac{\frac{7}{18}}{\frac{7}{18} + \frac{3}{14}} = ?$$

BAYES' THEOREM Suppose that E and F are events from a sample space S such that $p(E) \neq 0$ and $p(F) \neq 0$. Then

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})}.$$

Definition 2.4.1: The **expected value**, also called the expectation or mean, of the random variable X on the sample space S is equal to

$$E(X) = \sum_{s \in S} p(s)X(s)$$

Example 2.4.2: A fair coin is flipped three times. Let S be the sample space of the eight possible outcomes, and let X be the random variable that assigns to an outcome the number of heads in this outcome. What is the expected value of X?

X	$P(X)$	$x\,P(x)$
3	$1/8$	$3/8$
2	$3/8$	$6/8$
1	$3/8$	$3/8$
0	$1/8$	0

$$E = \sum_x x\,P(x) = \frac{12}{8} = 1.5$$