1.3 Basis and Dimension

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Basis and Dimension

Definition 2.3.1 *Let $\mathbb{V}$ be a complex (real) vector space. $V \in \mathbb{V}$ is a* **linear combination** *of the vectors $V_0, V_1, \ldots, V_{n-1}$ in $\mathbb{V}$ if V can be written as*

$= v$

$$V = \underline{c_0} \cdot \underline{V_0} + \underline{c_1} \cdot \underline{V_1} + \cdots + \underline{c_{n-1}} \cdot \underline{V_{n-1}} \quad \text{for some } c_0, c_1, \ldots, c_{n-1} \text{ in } \mathbb{C}\ (\mathbb{R}).$$

Example 1: construct a vector $v \in \mathbb{V} = \mathbb{C}^3$ as a linear combination of the following vectors:

$$v_1 = \begin{bmatrix} i \\ 2 \\ -1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2+i \\ 3 \end{bmatrix}, v_3 = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$
$$= v_1 + v_2 + v_3 = \begin{bmatrix} 4+i \\ 2+i \\ 3 \end{bmatrix}$$

Definition 2.3.2 *A set $\{V_0, V_1, \ldots, V_{n-1}\}$ of vectors in $\mathbb{V}$ is called* **linearly independent** *if*

$$\mathbf{0} = c_0 \cdot V_0 + c_1 \cdot V_1 + \cdots + c_{n-1} \cdot V_{n-1}$$

implies that $c_0 = c_1 = \cdots = c_{n-1} = 0$. This means that the only way that a linear combination of the vectors can be the zero vector is if all the c_j are zero.

Example 2: Show that the following vectors are linearly independent

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\begin{bmatrix} c_1 \\ c_1 + c_2 \\ c_1 + c_2 + c_3 \end{bmatrix} = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$\boxed{c_1 = 0}$

$c_1 + c_2 = 0 \Rightarrow \boxed{c_2 = 0}$

$c_1 + c_2 + c_3 = 0 \Rightarrow \boxed{c_3 = 0}$

$1 \ 0$

$2^1 + 2^0 = 2$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \to 0 \\ \to 1 \\ \to 2 \\ \to 3 \end{matrix}$

$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{C}^4$

$|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$|10\rangle \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$|11\rangle$

Example 3: Show that the following vectors are not linearly independent (**linear dependent**)

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

$$\begin{bmatrix} c_1 + 2c_3 \\ c_1 + c_2 - c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$c_1 + 2c_3 = 0 \Rightarrow c_1 = -2c_3$

$c_1 + c_2 - c_3 = 0$ if $\boxed{c_3 = 1} \Rightarrow \boxed{c_1 = -2}$

$$\begin{bmatrix} c_1 + 2c_3 \\ c_1 + c_2 - c_3 \\ c_1 + c_2 - c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} c_1 + 2c_3 = 0 \Rightarrow c_1 = -2c_3 \\ c_1 + c_2 - c_3 = 0 \quad \text{if } \boxed{c_3 = 1} \Rightarrow \boxed{c_1 = -2} \\ c_1 + c_2 - c_3 = 0 \quad \boxed{c_2 = -3} \end{matrix}$$

Definition 2.3.3 *A set $\mathcal{B} = \{V_0, V_1, \ldots, V_{n-1}\} \subseteq \mathbb{V}$ ($\mathbb{C}^n$) of vectors is called a* **basis** *of a (complex) vector space $\mathbb{V}$ if both*

$v = c_0 v_0 + c_1 v_1 + \cdots + c_{n-1} v_{n-1} = \sum_{i=0}^{n-1} c_i v_i$

(i) *every, $V \in \mathbb{V}$ can be written as a linear combination of vectors from $\mathcal{B}$ and*

(ii) *$\mathcal{B}$ is linearly independent.* $\Rightarrow \forall c_1 \quad c_0 v_0 + c_1 v_1 + c_2 v_2 + \cdots + c_{n-1} v_{n-1} = 0 \Rightarrow \forall c_i = 0$

Standard Basis

■ $\mathbb{C}^n$ (and $\mathbb{R}^n$):

c_0

$$E_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, E_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \leftarrow i, \ldots, E_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \leftarrow n-1.$$

Note: Every vector $v = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{V} = \mathbb{C}^n$ can be written as follows

$$v = \sum_{i=0}^{n-1} c_i E_i = c_0 E_0 + c_1 E_1 + \cdots + c_{n-1} E_{n-1}$$

$$= \begin{bmatrix} c_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

The basis for the vector space $\mathbb{C}^{m \times n}$ consists of matrices of the form

$$E_{j,k} = \begin{array}{c} \\ \mathbf{0} \\ \mathbf{1} \\ \vdots \\ \boldsymbol{j} \\ \vdots \\ \mathbf{m-1} \end{array} \begin{array}{c} \begin{array}{ccccccc} \mathbf{0} & \mathbf{1} & \cdots & \boldsymbol{k} & \cdots & \mathbf{n-1} \end{array} \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \end{array}.$$

k column   j row

$$E_{0,0} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

Any m-by-n matrix, A can be written as the sum:

$$A = \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \underbrace{A[j,k]}_{C(j,k)} \cdot E_{j,k}.$$

Definition 2.3.4 *The* **dimension** *of a (complex) vector space is the number of elements in a basis of the vector space.*

- In general, $\mathbb{R}^n$ has dimension n as a real vector space.
- $\mathbb{C}^n$ has dimension n as a complex vector space.
- $\mathbb{C}^{m\times n}$: the dimension is mn as a complex vector space.
- The dimension of $\mathbb{V} \times \mathbb{V}'$ is the dimension of $\mathbb{V}$ plus the dimension of $\mathbb{V}'$.

$\mathbb{C}^4$

Example 4: a) Find the coefficient of the vector $v = \begin{bmatrix} 7 \\ -17 \end{bmatrix}$ with respect to the basis

$$B = \left\{ \overset{v_1}{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} -2 \\ 4 \end{bmatrix}} \right\}.$$

b) Find the coefficient of the vector v with respect to the basis $D = \left\{ \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \end{bmatrix} \right\}$?

$$v = c_1 v_1 + c_2 v_2$$

$$\begin{bmatrix} 7 \\ -17 \end{bmatrix} = \begin{bmatrix} c_1 - 2c_2 \\ -3c_1 + 4c_2 \end{bmatrix}$$

$$\begin{cases} c_1 - 2c_2 = 7 \quad (2) \\ -3c_1 + 4c_2 = -17 \end{cases}$$

$$-c_1 = -3 \Rightarrow \boxed{c_1 = 3}$$

$$-2c_2 = 4$$

$$3 - 2c_2 = 7$$

$$\boxed{c_2 = -2}$$