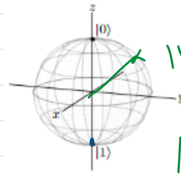
Postulate 1:

Definition 4.1.1: A quantum bit or a **qubit** is a unit of information describing a two dimensional quantum system

> "*Associated to any isolated physical system is a complex vector space with inner product (i.e. a Hilbert space) known as the state space of the system. The system is completely described by its state vector, which is a unit vector in the system's state space.*" (Nielsen and Chuang).

$|1\rangle$

"*ket*" notation $|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}$ $|1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

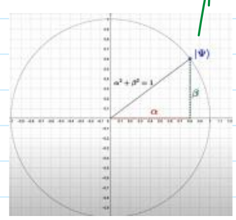
Classical 5- bit-string 10111 $\longrightarrow$ Multi-Quantum bit-2^5 $\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$

$|\varphi\rangle = \alpha_1|0\ldots\rangle + \alpha_2|0\ldots1\rangle$ @ n

$\sum_{i=1}^{n} |\alpha_i|^2 = 1$

Superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,\quad \alpha, \beta \in \mathbb{C}$

$|\alpha|^2 + |\beta|^2 = 1$



Note 4.1.1: Any nonzero element of $\mathbb{C}^2$ can be converted into a qubit.

Example 4.1.1: Write the vector $v = \begin{bmatrix}5+3i\\6i\end{bmatrix}$ as the sum of the qubits $|0\rangle$ *and* $|1\rangle$

$$|v| = \sqrt{25+9+36} = \sqrt{70}$$

$$|v\rangle = \frac{5+3i}{\sqrt{70}}|0\rangle + \frac{6i}{\sqrt{70}}|1\rangle$$

"***bra***" notation: $\langle\psi| = |\psi\rangle^\dagger$

<u>Polar Representation of Qubit</u>

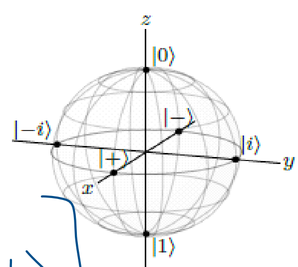
$e^{i\varphi_\alpha} \cdot e^{-i\varphi_\alpha}$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$= \cancel{\rho_\alpha e^{i\phi_\alpha}}|0\rangle + \rho_\beta e^{i\phi_\beta}|0\rangle$

$0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi$

$= e^{i\varphi_\alpha}\left[\rho_\alpha|0\rangle + \rho_\beta e^{i(\varphi_\beta - \varphi_\alpha)}|1\rangle\right]$

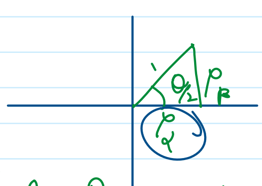


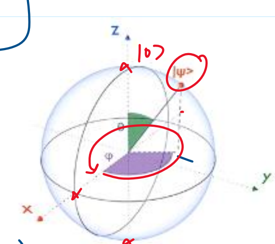
$|\alpha|^2 + |\beta|^2 = 1$

$\rho_\alpha^2 + \rho_\beta^2 = 1$

$|\psi\rangle = e^{i\varphi_\alpha}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle\right]$

Global.





$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle$

Let $\frac{\theta}{2}$ be angle between ρ_α & ρ_β

θ: determine how state at $|0\rangle$ or $|1\rangle$

θ: determine how state at $|0\rangle$ or $|1\rangle$

φ: phase rotation.

① Z-bases $\{|0\rangle, |1\rangle\}$

$\theta = 0$, φ = anything $\varphi = 0$.

$|\psi\rangle = |0\rangle$

$\theta = \pi$ $\varphi = 0$

$\boxed{|\psi\rangle = |1\rangle}$

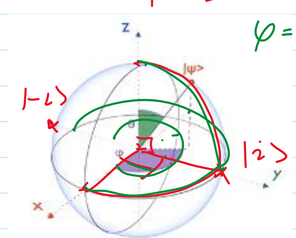
$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi}|1\rangle$

② $X = \{|+\rangle, |-\rangle\}$

$|i\rangle$ $|-i\rangle$

$\theta = \pi/2$ $\varphi = 3\pi/2$

$|-i\rangle$ $|i\rangle$



$\theta = \frac{\pi}{2}$, $\varphi = 0$

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$

$\theta = \frac{\pi}{2}$, $\varphi = \pi$

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = |-\rangle$

③ $Y = \{|i\rangle, |-i\rangle\}$

$\theta = \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle = |i\rangle$

$\theta = \frac{\pi}{2}$, $\varphi = \frac{3\pi}{2}$

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}(-i)|1\rangle = |-i\rangle$

<u>Quantum Gates</u>

Postulate 2: How qubit(s) transform.

"The evolution of a closed quantum system is described by a unitary transformation. That is, the state |ψ> of the system at time t_1 is related to the state of |ψ'> of the system at time t_2 by a unitary operator U which

depends only on times t_1 and t_2." (Nielsen and Chuang).

$$U|\psi\rangle = |\psi'\rangle$$

Definition 4.1.2: A quantum gate is simply an operator that acts on qubits. Such operators will be represented by unitary matrices. The essential properties of quantum logic gates:

- $U^\dagger$ is unitary.
- U^{-1} is unitary.
- $U^{-1} = U^\dagger$ (which is the criterion for determining unitarity).
- $U^\dagger U = \mathbb{1}$
- $|\det(U)| = 1$.
- The columns (rows) of U form an orthonormal set of vectors.

Example 4.1.2: Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, compute $U|0\rangle$ and $U|1\rangle$

$$U|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} = |+\rangle$$

Example 4.1.3: Consider $U|0\rangle = \frac{\sqrt{2}-i}{2}|0\rangle - \frac{1}{2}|1\rangle$, $U|1\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{2}+i}{2}|1\rangle$, and let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be a quantum state

Compute $U|\psi>$ and check if U is a valid quantum gate

$$U|\psi\rangle = \alpha U|0\rangle + \beta U|1\rangle$$
$$= \left(\frac{\sqrt{2}-i}{2}\alpha|0\rangle - \frac{\alpha}{2}|1\rangle\right) + \left(\frac{\beta}{2}|0\rangle + \frac{\sqrt{2}+i}{2}\beta|1\rangle\right)$$
$$= \left(\frac{\sqrt{2}-i}{2}\alpha + \frac{\beta}{2}\right)|0\rangle + \left(-\frac{\alpha}{2} + \frac{\sqrt{2}+i}{2}\beta\right)|1\rangle$$

$$U = \begin{bmatrix}\frac{\sqrt{2}-i}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{\sqrt{2}+i}{2}\end{bmatrix} \qquad U^\dagger U \overset{?}{=} I$$

Note 4.1.2: 1) Quantum gates are linear maps that keep the total probability equal to 1.
2) Quantum gates are unitary metrices and unitary matrices are quantum gates.

Example 4.1.4: Determine whether the following gate is quantum gate.

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}$$

$$UU^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & i\\ -i & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 2i\\ -2i & 2\end{bmatrix} \neq I$$

$$UU^{\dagger} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 2i \\ -2i & 2 \end{bmatrix} \neq I$$

Common One-Qubit Quantum Gates

1. The identity gate: $I|0\rangle = |0\rangle$, $I|1\rangle = |1\rangle$ $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I|0\rangle = |0\rangle,\ I|1\rangle = |1\rangle.$

2. The Pauli X gate, or NOT gate:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X|0\rangle = |1\rangle,\ X|1\rangle = |0\rangle.$$

Example 4.1.5: Compute $X|+\rangle,\ X|-\rangle,\ X|i\rangle,\ X|-i\rangle$

$$X|-i\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{-i}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = -i|i\rangle$$

$e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$\theta = \frac{\pi}{2} \quad \varphi = \pi/2$

$$|i\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}i|1\rangle = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ i \end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

3. The Pauli Z gate: $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Example 4.1.6: Compute $Z|0\rangle,\ Z|1\rangle, Z|+\rangle,\ Z|-\rangle,\ Z|i\rangle,\ Z|-i\rangle$

$$Z|i\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = |-i\rangle$$

4. The Pauli Y gate: $Y = iXZ$ $\qquad I^2 = X^2 = Y^2 = Z^2 = -iXYZ = I$

$$Y = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

5. The **Hadamard gate** H: $\qquad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$

Example 4.1.7: Compute $H|0\rangle,\ H|1\rangle, H|+\rangle,\ H|-\rangle,\ H|i\rangle,\ H|-i\rangle$

Example 4.1.7: Compute $H|0>, \ H|1>, H|+>, \ H|->, \ H|i>, \ H|-i>$

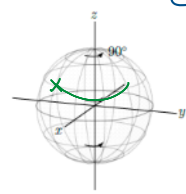
$$H|-i\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1-i\\ 1+i\end{pmatrix} = \frac{1-i}{2}|0\rangle + \frac{1+i}{2}|1\rangle$$

$$|\psi\rangle \longrightarrow \boxed{U} \longrightarrow |\varphi\rangle$$

6. **Phase gate**, which is the square root of the Z gate (i.e., $S^2 = Z$):

$$S = \begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix} \quad e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \qquad \begin{bmatrix}1 & 0\\ 0 & e^{i\pi/2}\end{bmatrix} \qquad S^4 = I$$

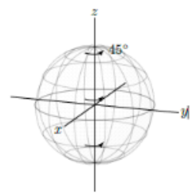
S is a rotation about the z-axis by 90 ∘



7. T gate, which is called $\frac{\pi}{8}$ (or quarter of Z gate):

$$T = \begin{bmatrix}1 & 0\\ 0 & exp(\frac{i\pi}{4})\end{bmatrix}$$

T is a rotation about the z-axis by 45 ∘ $= \frac{\pi}{4}$



$T^2 = S$ and $T^4 = Z$,

Gate	Action on Computational Basis	Matrix Representation
Identity	$I\|0\rangle = \|0\rangle$ $I\|1\rangle = \|1\rangle$	$I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
Pauli X	$X\|0\rangle = \|1\rangle$ $X\|1\rangle = \|0\rangle$	$X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
Pauli Y	$Y\|0\rangle = i\|1\rangle$ $Y\|1\rangle = -i\|0\rangle$	$Y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$
Pauli Z	$Z\|0\rangle = \|0\rangle$ $Z\|1\rangle = -\|1\rangle$	$Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$
Phase S	$S\|0\rangle = \|0\rangle$ $S\|1\rangle = i\|1\rangle$	$S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$
T	$T\|0\rangle = \|0\rangle$ $T\|1\rangle = e^{i\pi/4}\|1\rangle$	$T = \begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}$
Hadamard H	$H\|0\rangle = \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$ $H\|1\rangle = \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

Example 4.1.8: Compute $HSTH|0\rangle$

$$= HST|+\rangle = \frac{1}{\sqrt{2}} HS(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle)$$

$$|0\rangle - \boxed{H} - \boxed{T} \rightarrow \boxed{S} - \boxed{H} \quad = \frac{1}{\sqrt{2}} H\left(|0\rangle + i\, e^{i\pi/4}|1\rangle\right) \qquad i = e^{i\pi/2} \qquad e^{i\frac{\pi}{2}}$$

$$= \frac{1}{\sqrt{2}} H\left(|0\rangle + e^{i3\pi/4}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\underbrace{|+\rangle}_{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)} + e^{i3\pi/4}\underbrace{|-\rangle}_{\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)}\right)$$

$$= \frac{1}{\cdot}\left((1 + e^{i3\pi/4})|0\rangle + (1 - e^{\frac{i3\pi}{4}})|1\rangle\right)$$

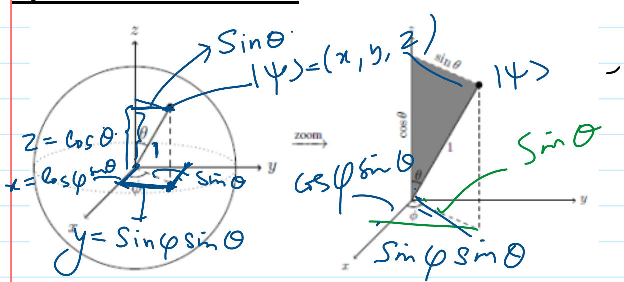
$$= \frac{1}{2}\left(\left(1+e^{i3\pi/4}\right)|0\rangle + \left(1-e^{i3\pi/4}\right)|1\rangle\right)$$

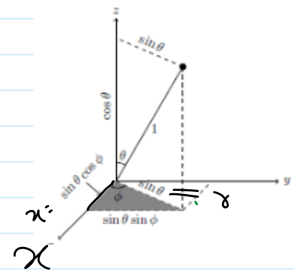
Example 4.1.9: Compute $X^{51}H^{97}T^{36}Z^{25}|0>$

$$\underbrace{(T^4)^9 (Z)^9}_{=Z} \qquad = XHZZ|0\rangle \qquad X\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right) = \frac{1}{\sqrt{2}}(|1\rangle+|0\rangle)$$

$$= XH|0\rangle = X|+\rangle = |+\rangle$$

Spherical Coordinates





$x = sin\theta cos\varphi$
$y = sin\theta sin\varphi$
$z = \cos\theta$

Example 4.1.10: Write the following qubit state in terms of (θ, φ), and then find the point in the Block sphere

$$\frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$r = \sqrt{3/4 + 1/4} = 1$$

$$\varphi = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi/6$$

$$|\psi\rangle = \frac{3+i\sqrt{3}}{4}|0\rangle - \frac{1}{2}|1\rangle$$

$$= \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}+i}{2}\right)|0\rangle - \frac{1}{2}|1\rangle$$

$$= \frac{\sqrt{3}}{2}e^{i\pi/6}|0\rangle + \frac{1}{2}e^{i\pi}|1\rangle$$

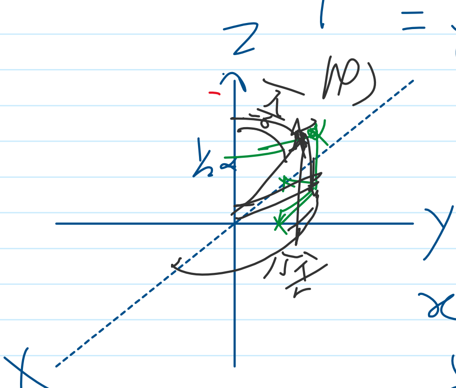
$$= e^{i\pi/6}\left(\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}e^{i5\pi/6}|1\rangle\right)$$

$$\sin\frac{\theta}{2} = \frac{1}{2}$$

$$\frac{\theta}{2} = \frac{\pi}{6} \Rightarrow \theta = \frac{2\pi}{6}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \boxed{\theta = \pi/3}$$

$$\frac{e^{i\pi}}{e^{i\pi/6}} = e^{i\pi - i\pi/6} = e^{i5\pi/6}$$



$$\theta = \frac{\pi}{3} \qquad \varphi = \frac{5\pi}{6}$$

$$x = \cos\varphi \sin\theta = -\frac{\sqrt{3}}{2}\,\frac{\sqrt{3}}{2} = -3/4$$

$$y = \sin\varphi \sin\theta = 1/2 \; \sqrt{3}/2 = \sqrt{3}/4$$

$$z = \cos\theta = 1/2$$

Exercise 2.28. Consider the gate $R_z(\theta)$, which rotates about the z-axis by angle θ:

$$R_z(\theta)|0\rangle = |0\rangle,$$
$$R_z(\theta)|1\rangle = e^{i\theta}|1\rangle.$$

The Z gate, S gate, and T gate are all specific instances of the R_z gate, with $Z = R_z(\pi)$, $S = R_z(\pi/2)$, and $T = R_z(\pi/4)$. Say $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a normalized quantum state, i.e., $|\alpha|^2 + |\beta|^2 = 1$.

(a) Calculate $R_z(\theta)|\psi\rangle$.

(b) Show that the total probability of $R_z(\theta)|\psi\rangle$ is 1, so $R_z(\theta)$ is a valid quantum gate, and hence, Z, S, and T are all valid quantum gates.

$$|\psi\rangle = \boxed{\frac{3+i\sqrt{3}}{4}}|0\rangle - \boxed{\frac{1}{2}}|1\rangle \equiv \cos\left(\frac{\theta}{2}\right)|0\rangle + \boxed{\sin\left(\frac{\theta}{2}\right)} e^{i\varphi}|1\rangle$$

$\sin\frac{\theta}{2} = \frac{1}{2}$

$\frac{\theta}{2} = \frac{\pi}{6}$

$\cos\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2}$

$$\boxed{\theta = \frac{2\pi}{6} = \frac{\pi}{3}}$$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$-1 = \cos(\pi) + i\sin(\pi) = e^{i\pi}$

$$= \frac{\sqrt{3}}{2}\left(\boxed{\frac{\sqrt{3}+i}{2}}\right)|0\rangle \boxed{-} \boxed{\frac{1}{2}}|1\rangle$$

$$= \cos\left(\frac{\theta}{2}\right)\left(\frac{\sqrt{3}+i}{2}\right)|0\rangle + \sin\left(\theta/2\right) e^{i\pi}|1\rangle$$

$$= \cos\left(\frac{\theta}{2}\right) e^{i\pi/6}|0\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\pi}|1\rangle$$

$$= e^{i\pi/6}\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\theta/2\right) e^{i5\pi/6}|1\rangle\right)$$

Globale.

$\frac{e^{i\pi}}{e^{i\pi/6}} = e^{i(\pi - \pi/6)}$

$\frac{\sqrt{3}+i}{2} = 1e^{i\frac{\pi}{6}}$

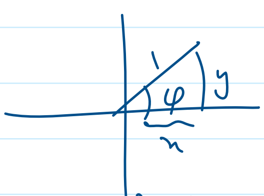
$= re^{i\gamma}$

$r = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$

$\gamma = \tan^{-1}\left(\frac{\frac{1}{2}}{\sqrt{3}/2}\right)$

$\gamma = \frac{\pi}{6}$

$$\varphi = \frac{5\pi}{6}, \quad \theta = \frac{\pi}{3}$$



$x = \cos\varphi$

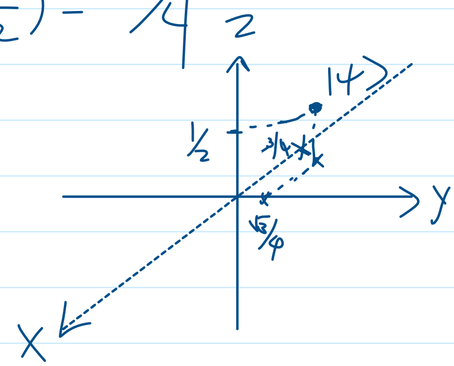
$y = \sin\varphi$

$x = \cos\varphi \sin\theta = \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = -3/4$

$y = \sin\varphi \sin\theta = \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}/4$

$z = \cos\theta = \frac{1}{2}$

$$|\psi\rangle = \left(-3/4, \sqrt{3}/4, \frac{1}{2}\right)$$



4.2 Measuring and Combing Quantum States

Tuesday, October 14, 2025 12:01 PM

Postulate 3: The effect of measurement.

"Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi>$ immediately before the measurement then the probability that result m occurs is given by:

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$$

and the state of the system after measurement is:

$$\frac{M_m \, \psi\rangle}{\sqrt{\langle \psi \, M_m^\dagger M_m \, \psi \rangle}}$$

The measurement operators satisfy the *completeness equation*:

Normalization

$$\sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle = I$$

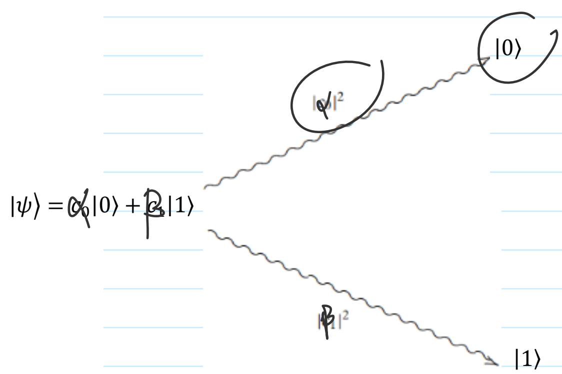
The completeness equation expresses the fact that probabilities sum to one:

$$1 = \sum_m p(m) = \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle$$ "

Some important measurement operators are $M_0 = \underline{|0\rangle\langle 0|}$ and $M_1 = \underline{|1\rangle\langle 1|}$

$$M_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1, 0] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0, 1] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$



Observe that $\underline{M_0^\dagger M_0} + \underline{M_1^\dagger M_1} = I$ and are thus complete.

Example:

$$|\psi\rangle = a|0\rangle + b|1\rangle$$
$$p(0) = \langle \psi | M_0^\dagger M_0 | \psi \rangle$$

Note that $M_0^\dagger M_0 = M_0$, hence

$$p(0) = \langle \psi | M_0 | \psi \rangle = [a^*, b^*] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} =$$
$$= [a^*, b^*] \begin{bmatrix} a \\ 0 \end{bmatrix} = |a|^2$$

Hence the probability of measuring $|0\rangle$ is related to its probability amplitude a by way of $|a|^2$.

$\langle m|\psi\rangle$ *is the amplitude of* $|\psi\rangle$ *with respect to* $\langle m|$ $\quad \langle 0|\psi\rangle$

$\langle m|\psi\rangle$ is the *projection* $\langle \psi|$ of onto $\langle m|$

Measurement in the Z-Basis

Example 4.2.1: Measure the following with getting $|0\rangle$ and $|1\rangle$

A) $\underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/6}|1\rangle\right)}_{}$ $\;|\psi\rangle$ B) $\frac{2}{3}|0\rangle + \frac{1-2i}{3}|1\rangle$ $\;|\varphi\rangle$

$$p(0) = \left|\langle 0|\psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\overbrace{\langle 0|0\rangle}^{=1} + e^{i\pi/6}\overbrace{\langle 0|1\rangle}^{=0}\right)\right|^2 = \frac{1}{2}$$

$$p(1) = \left|\langle 1|\psi\rangle\right|^2 = \left|\frac{1}{\sqrt{2}}\left(\overbrace{\langle 1|0\rangle}^{=0} + e^{i\pi/6}\overbrace{\langle 1|1\rangle}^{=1}\right)\right|^2 = \left|\frac{1}{\sqrt{2}}e^{i\pi/6}\right|^2 = \frac{1}{2}\left|\overbrace{e^{i\pi/6}e^{-i\pi/6}}^{=1}\right|^2 = \frac{1}{2}$$

Example 4.2.2: Find the value of A of the given qubit. $A\left(\sqrt{2}|0\rangle+i|1\rangle\right)$. $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$

$$|A\sqrt{2}|^2+|Ai|^2=1 \qquad |\alpha|^2+|\beta|^2=1$$

$$2|A|^2+|A|^2=1 \qquad |A|^2|i|^2$$

$$3\cdot|A|^2=1 \Rightarrow |A|=\frac{1}{\sqrt{3}} \qquad =|A|^2(i)(-i)=|A|^2$$

$$A=\frac{1}{\sqrt{3}}$$

Measurement in Other Bases

$$|+\rangle=\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

Example 4.2.3: Measure the following qubit with getting $\{|+\rangle,|-\rangle\}$, and $\{|i\rangle,|-i\rangle\}$

$$|\psi\rangle=\frac{\sqrt{3}}{2}|0\rangle+\frac{1}{2}|1\rangle$$

$$\langle+|\psi\rangle=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$

$$\left(\frac{1}{\sqrt{2}}\langle0|+\frac{1}{\sqrt{2}}\langle1|\right)\left(\frac{\sqrt{3}}{2}|0\rangle+\frac{1}{2}|1\rangle\right)$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$P_+=|\langle+|\psi\rangle|^2=\left|\frac{\sqrt{3}+1}{2\sqrt{2}}\right|^2$$

$$\frac{(\sqrt{3}+1)(\sqrt{3}+1)}{8}=\frac{4+2\sqrt{3}}{8}$$

$$=\frac{2+\sqrt{3}}{4}$$

$$P_-=\frac{2-\sqrt{3}}{4} \qquad P_++P_-=\frac{2+\sqrt{3}}{4}+\frac{2-\sqrt{3}}{4}=\frac{4}{4}=1$$

$$|i\rangle=\frac{1}{\sqrt{2}}(|0\rangle+i|1\rangle)=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix}$$

$$P_i=|\langle i|\psi\rangle|^2 \qquad \langle i|\psi\rangle=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{\pm i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{pmatrix}$$

$$=\left|\frac{\sqrt{3}-i}{2\sqrt{2}}\right|^2=\frac{4}{8}=1/2 \qquad =\frac{\sqrt{3}\pm i}{2\sqrt{2}}$$

Example 4.2.4: Consider the following states, which will call $|a\rangle$ *and* $|b\rangle$:

$$\langle a|b\rangle=0$$

$$|a\rangle=\frac{\sqrt{3}}{2}|0\rangle+\frac{i}{2}|1\rangle$$
$$|b\rangle=\frac{i}{2}|0\rangle+\frac{\sqrt{3}}{2}|1\rangle$$

$$\langle a|\psi\rangle=\left(\frac{\sqrt{3}}{2}\langle0|-\frac{i}{2}\langle1|\right)\left(\frac{\sqrt{3}}{2}|0\rangle+\frac{1}{2}|1\rangle\right)$$

$$=\frac{3-i}{4}$$

Measure the qubit $|\psi\rangle=\frac{\sqrt{3}}{2}|0\rangle+\frac{1}{2}|1\rangle$ in qubits $|a\rangle$ and $|b\rangle$

$$\langle b|\psi\rangle=\frac{-\sqrt{3}i+\sqrt{3}}{4}$$

$$P=|\langle a|\psi\rangle|^2 \quad |3-i|^2 \quad 10 = 5$$

Measure the qubit $|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$ in qubits $|a\rangle$ and $|b\rangle$

$$\langle b|\psi\rangle = \frac{\quad}{4}$$

$$P_a = |\langle a|\psi\rangle|^2 = \left|\frac{3-i}{4}\right|^2 = \frac{10}{16} = \frac{5}{8}$$

$$P_b = |\langle b|\psi\rangle|^2 = \left|\frac{\sqrt{3}-\sqrt{3}i}{4}\right|^2 = \frac{6}{16} = 3/8$$

Postulate 4: How qubits combine together into systems of qubits.

> *"The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Suppose systems* 1 *through* n *and system* i *is in state* $|\psi_i>$, *then the joint state of the total system is* $|\psi_1>\otimes|\psi_2>\cdots\otimes|\psi_n>$.

Definition 4.2.1: (Tensor Product)
Let's imagine that we have two quantum states A and B. We know that we can describe the state of A as $|\psi\rangle_A \in \mathbb{C}^{d_1}$ and the one of B as $|\psi\rangle_B \in \mathbb{C}^{d_2}$. We can combine the state $|\psi\rangle_{AB} \in \mathbb{C}^{d_1\times d_2}$ using Tensor product which is defined as:

$$|\psi\rangle_A = \alpha_A|0\rangle + \beta_A|1\rangle \text{ and } |\psi\rangle_B = \alpha_B|0\rangle + \beta_B|1\rangle$$

$$\begin{bmatrix}\alpha_A\\ \beta_A\end{bmatrix} \otimes \begin{bmatrix}\alpha_B\\ \beta_B\end{bmatrix}$$

$$|\psi\rangle_{AB} = |\psi\rangle_A \bigotimes |\psi\rangle_B = \begin{bmatrix}\alpha_A|\psi\rangle_B\\ \beta_A|\psi\rangle_B\end{bmatrix} = \begin{bmatrix}\alpha_A\alpha_B\\ \alpha_A\beta_B\\ \beta_A\alpha_B\\ \beta_A\beta_B\end{bmatrix}$$

$$\begin{bmatrix}\alpha_A \begin{bmatrix}\alpha_B\\ \beta_B\end{bmatrix}\\ \beta_A \begin{bmatrix}\alpha_B\\ \beta_B\end{bmatrix}\end{bmatrix}$$

Example 4.2.5: Find all combination of tensor product of the states $|0\rangle$ *and* $|1\rangle$

binary decimal

$$|0\rangle \otimes |1\rangle = |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \longrightarrow \text{position } 1 \quad (01 \cong 1$$

$$\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Example 4.2.6: Compute $|+\rangle|1\rangle$

$$\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

<u>Outer Products:</u> Let $|\psi\rangle_A = \alpha_A|0\rangle + \beta_A|1\rangle$ and $|\psi\rangle_B = \alpha_B|0\rangle + \beta_B|1\rangle$ bet two quantum states, the outer product defined as:

$$|\psi\rangle_A\langle\psi|_B = \underbrace{\underbrace{\begin{pmatrix}\boxed{\alpha_A}\\ \beta_A\end{pmatrix}}_{2\times1}\underbrace{\begin{pmatrix}\boxed{\alpha_B^*} & \boxed{\beta_B^*}\end{pmatrix}}_{1\times2}}_{2\times2}$$

$$|\psi\rangle_A\langle\psi|_B = \begin{bmatrix}\alpha_A\alpha_B^* & \alpha_A\beta_B^*\\ \beta_A\alpha_B^* & \beta_A\beta_B^*\end{bmatrix}$$

Example 4.2.7: Find a) $M = |0\rangle\langle 1| + |1\rangle\langle 0|, N = |i\rangle\langle +|$, Is M *and* N *represent* valid quantum gates (check Hermitian and unitary)?

Example 4.2.7: Find a) $M = |0\rangle\langle 1| + |1\rangle\langle 0|, N = |i\rangle\langle +|$, Is M *and* N *represent* valid quantum gates (check Hermitian and unitary)?

$$M = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

M is unitary $(M = X)$

$M = M^\dagger$ is Hermitian

$$N = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & i \end{pmatrix}$$

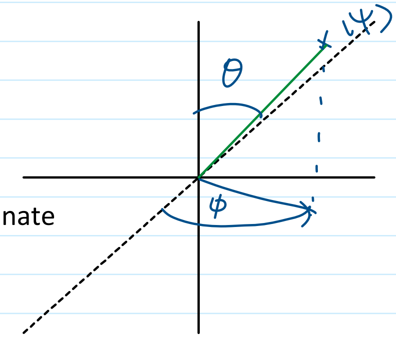
$NN^\dagger \neq I$ neither unitary nor Hermitian.

4.3 Rotations on the Bloch Sphere

Wednesday, October 15, 2025 2:31 PM

A single qubit state can be visualized as a point on the Bloch sphere:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$



Quantum **rotation gates** correspond to *rotating this vector* around one of the coordinate axes X,Y, or Z.

The Three Basic Rotation Gates:

Each rotation gate is defined as an exponential of a **Pauli matrix**:

$$R_n(\theta) = e^{-i\frac{\theta}{2}\sigma_n}, \quad n \in \{x, y, z\}$$

$$\cos\left(\frac{\theta}{2}\right) I + i \sin\left(\frac{\theta}{2}\right) \sigma_n .$$

where:

- $\sigma_x, \sigma_y, \sigma_z$ are the **Pauli matrices**:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Rotation about X-axis: rotates the state around the X-axis by angle θ.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \cos\left(\frac{\theta}{2}\right) I + i \sin\left(\frac{\theta}{2}\right) \sigma_X = \begin{pmatrix} \cos\theta/2 & i\sin(\theta/2) \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

$$\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$R_X(\pi) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = e^{i\pi/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma = \pi/2 .$$

Global .

(b) Rotation about Y-axis rotates the state around the Y-axis by angle θ.

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \cos\left(\frac{\theta}{2}\right) I + i \sin\left(\frac{\theta}{2}\right) \sigma_y = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

(c) Rotation about Z-axis rotates the state around the Z-axis by angle θ.

Now, say there is a partial flip on the Bloch sphere, a rotation by angle θ about the axis $\hat{n} = (n_x, n_y, n_z)$ is given by

$$\text{Global} \leftarrow e^{i\gamma}\left[\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)(n_xX + n_yY + n_zZ)\right],$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

where γ is a global phase that we can set to anything (or drop), since it does not have any physical relevance

Example 4.3.1: Consider the Hadamard gate is a rotation by θ = 180 ∘ = π radians about the axis halfway between the x- and z-axes, express the Hadamard using *U*.

$n_x = n_z \qquad n_y = 0$

$$n_x^2 + \underbrace{n_y^2}_{=0} + n_z^2 = 1$$

$$2n_x^2 = 1 \Rightarrow n_x = \frac{1}{\sqrt{2}}.$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \cos\left(\frac{\theta}{2}\right) I + i \sin\left(\frac{\theta}{2}\right) \cdot \frac{1}{\sqrt{2}}\left(\sigma_x + \sigma_z\right)$$

$\theta = \pi$

$$= \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} I + i \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= i \, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot = \underline{\underline{H}}$$

Global