7.1 Information and Shannon Entropy

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- **Entropy** is a way to quantify the information content in a signal.
- Entropy is a measure of unpredictability or information content.
- In cryptography, the strength of encryption keys and passwords depends on their unpredictability, which is measured by entropy. High entropy ensures that keys and passwords are resistant to brute-force attacks.

Example 7.1.1: Alice and Bob are exchanging the following messages in below table. Let us say that Alice can only send one of four different messages coded by the letters A, B,C, and D. Bob is a careful listener, so he keeps track of the frequency of each letter.

Symbol	Meaning
A	"I feel sad now"
B	"I feel angry now"
C	"I feel happy now"
D	"I feel bored now"

By observing N consecutive messages from Alice, he reports the following:

A appeared N_A times

B appeared N_B times

C appeared N_C times

D appeared N_D times

$$P(A) = \frac{N_A}{N}$$

Compute the probabilities for each letter based on Bob's observations.

$$P(A) = \frac{N_A}{N}$$

$$P(B) = \frac{N_B}{N}$$

$$P(C) = \frac{N_C}{N}$$

$$P(D) = \frac{N_D}{N}.$$

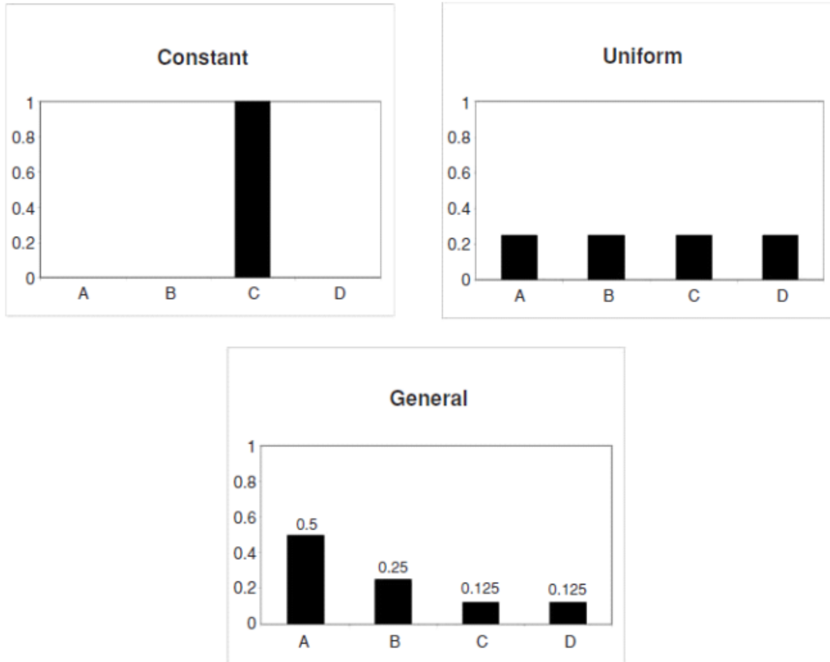
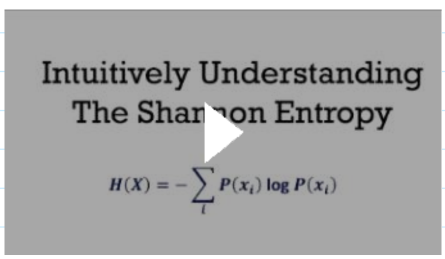


Figure 10.1. Three possible probability distributions.

[Intuitively Understanding the Shannon Entropy]



Definition 7.1.1 The **Shannon entropy** of a source with probability distribution $\{p_i\}$ is the quantity

$$H_S = -\sum_{i=1}^{n} p_i \times \log_2(p_i) = \sum_{i=1}^{n} p_i \times \log_2\left(\frac{1}{p_i}\right),$$

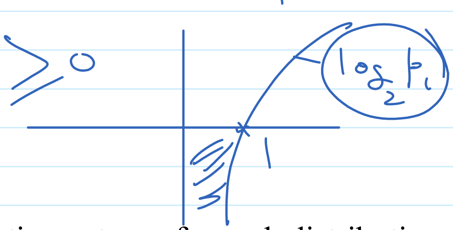
$-\log_2 p_i = \log_2 p_i^{-1}$

$= \log_2\left(\frac{1}{p_i}\right)$

Note 7.1.1: Shannon entropy is always positive or zero.

$0 \le p_i \le 1$

$H_S = -\sum p_i \log_2 p_i \geq 0$



Example 7.1.2: In Ex 7.1.1 compute the Shannon information entropy for each distributions

① Constant: $H_S = -\sum_{i=1}^{3} p_i \log_2 p_i$

$p_1 = P(A) = P(B) = P(D) = 0$

$p_3 = P(C) = 1$

$= -P(C) \log_2 P(C)$

$$= -1 \log_2(1) = 0$$

② Uniform: $H_S = -\sum_{i=1}^{4} (0.25) \log_2 (0.25)$

$$= -4(0.25) \log_2(0.25)$$

$$= 2$$

③ General: $H_S = -\left\{ \frac{1}{2} \log_2\left(\frac{1}{2}\right) + \frac{1}{4} \log_2\left(\frac{1}{4}\right) + \frac{1}{8} \log_2\left(\frac{1}{8}\right) + \frac{1}{8} \log_2\left(\frac{1}{8}\right) \right\}$

$\log_2 2^{-1} = -\cancel{\log_2 2}$ $\quad = -\left\{ -\frac{1}{2} - \frac{1}{2} - \frac{3}{8} - \frac{3}{8} \right\} = \frac{14}{8} = \frac{7}{4}$

7.2 Quantum Information and Von Neumann Entropy

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$$A \xrightarrow{|w_1\rangle\ |w_2\rangle \ \cdots\ |w_n\rangle} B$$

$$p(|w_1\rangle)=p_1, \ldots, \qquad P(|w_n\rangle)=p_n$$

The projection of $|v\rangle$ on $|w_i\rangle$

$$|w_i\rangle\langle w_i|\,|v\rangle$$

$$=\langle w_i|v\rangle\,|w_i\rangle$$

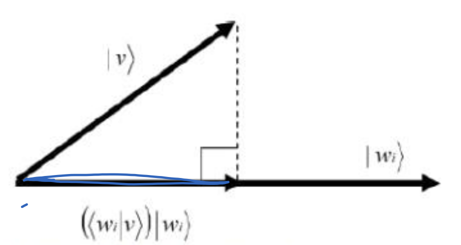

Figure 10.2. The projection of $|v\rangle$ onto $|w_i\rangle$.

$\therefore \langle w_i|v\rangle$ is the length of the projection

Definition 7.2.1 The **Density operator** is the weighted sum of a basic expression of the form $|\omega\rangle\langle\omega|$

$$D = p_1|w_1\rangle\langle w_1| + p_2|w_2\rangle\langle w_2| + \cdots + p_n|w_n\rangle\langle w_n|.$$

$$D=\sum_{i=1}^{n} p_i\,|w_i\rangle\langle w_i|$$

$$D(|v\rangle)=\sum_{i=1}^{n} p_i\,(\langle w_i|v\rangle)\,|w_i\rangle$$

Example 7.2.1: Let $|\omega_1\rangle = |+\rangle$ and $|\omega_2\rangle = |-\rangle$. Assume that $|\omega_1\rangle$ is sent with probability $p_1 = \frac{1}{4}$ and $|\omega_2\rangle$ is sent with probability $p_2 = \frac{3}{4}$. Describe the corresponding density matrix in the standard basis.

$$D(|0\rangle)=\sum_{i=1}^{2} p_i\,(\langle w_i|0\rangle)\,|w_i\rangle$$

$\langle +|0\rangle$

$=\frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)|0\rangle$

$$= p_1(\langle +|0\rangle)|+\rangle + p_2(\langle -|0\rangle)|-\rangle$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)|+\rangle + \frac{3}{4}\left(\frac{1}{\sqrt{2}}\right)|-\rangle$$

$\frac{1}{8}+\frac{3}{8}=\frac{4}{8}$ $\qquad$ $\frac{1}{8}-\frac{3}{8}=-\frac{2}{8}$

$$= \frac{1}{8}(|0\rangle+|1\rangle) + \frac{3}{8}(|0\rangle-|1\rangle) = \frac{1}{2}|0\rangle - \frac{1}{4}|1\rangle = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{4} \end{pmatrix}$$

$$D(|1\rangle) = \frac{1}{4}(\langle +|1\rangle)|+\rangle + \frac{3}{4}(\langle -|1\rangle)|-\rangle \quad \Big|\ \frac{1}{\sqrt{2}}(\langle 0| - \langle 1|)|1\rangle$$

$$D(|1\rangle) = \frac{1}{4}\left(\langle +|1\rangle\right)|+\rangle + \frac{3}{4}\left(\langle -|1\rangle\right)|-\rangle \quad \frac{1}{\sqrt{2}}(\underbrace{\langle 0|}_{0} - \langle 1|)|1\rangle$$

$$= \frac{1}{4\sqrt{2}}|+\rangle - \frac{3}{4\sqrt{2}}|-\rangle$$

$$= \frac{1}{8}(|0\rangle + |1\rangle) - \frac{3}{8}(|0\rangle - |1\rangle)$$

$$= -\frac{1}{4}|0\rangle + \frac{1}{2}|1\rangle = \begin{pmatrix} -1/4 \\ 1/2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1/2 & -1/4 \\ -1/4 & 1/2 \end{pmatrix} \qquad D = p_1|+\rangle\langle +| + p_2|-\rangle\langle -|)$$

Example 7.2.2: Write the density matrix of the following alphabet with associated probabilities:

$$\left\{|A\rangle = |0\rangle,\ |B\rangle = |1\rangle,\ |C\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle,\ |D\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right\}.$$

$$p(|A\rangle) = \frac{1}{2},\ p(|B\rangle) = \frac{1}{6},\ p(|C\rangle) = \frac{1}{6},\ \text{and } p(|D\rangle) = \frac{1}{6}.$$

$$D = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{6}|1\rangle\langle 1| + \frac{1}{6}|+\rangle\langle +| + \frac{1}{6}|-\rangle\langle -|$$

$|+\rangle\langle +|$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$|0\rangle\langle 0| + |0\rangle\langle 1|$

$+|1\rangle\langle 0| + |1\rangle\langle 1|)$

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{6} \end{pmatrix} + \begin{pmatrix} \frac{1}{12} & \frac{1}{12} \\ \frac{1}{12} & \frac{1}{12} \end{pmatrix} + \begin{pmatrix} \frac{1}{12} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{1}{12} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{6} & 0 \\ 0 & 4/12 \end{pmatrix} = \begin{pmatrix} 2/3 & 0 \\ 0 & 1/3 \end{pmatrix}$$

<u>Density operator acts on bra vectors</u>

$$\langle v|D = p_1\langle v|w_1\rangle\langle w_1| + p_2\langle v|w_2|\rangle\langle w_2 + \cdots + p_n\underline{\langle v|w_n\rangle}\langle w_n|.$$

$$\langle v|D|v\rangle = p_1\langle \underline{v|w_1}\rangle\langle \underline{w_1|v}\rangle$$

$$\langle v|D|v\rangle = p_1|\langle v|w_1\rangle|^2 + p_2|\langle v|w_2\rangle|^2 + \cdots + p_n|\langle v|w_n\rangle|^2. = \sum p_i |\langle v|w_i\rangle|^2.$$

Example 7.2.3: Assume Alice has a quantum alphabet consisting of only two symbols, the vectors

$$|w_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

and

$$|w_2\rangle = |0\rangle.$$

Assume that $|\omega_1\rangle$ is sent with probability $p_1 = \frac{1}{3}$ and $|\omega_2\rangle$ is sent with probability $p_2 = \frac{2}{3}$.
Bob uses the standard basis $\{|0\rangle, |1\rangle\}$ for his measurements. Compute $\langle 0|D|0\rangle$ and $\langle 1|D|1\rangle$.

$$D = \sum p_i |\omega_i\rangle\langle\omega_i|$$
$$= \frac{1}{3}|\omega_1\rangle\langle\omega_1| + \frac{2}{3}|\omega_2\rangle\langle\omega_2|$$
$$= \frac{1}{6}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & 1/6 \end{pmatrix}$$

$$\langle 0|D|0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}}_{1\times 2}\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{2\times 1} = \underline{\underline{\frac{5}{6}}}$$

$$\langle 1|D|1\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \underline{\underline{1/6}}$$

Note: If Alice is sending $|\omega_i\rangle$'s, from Bob's standpoint, the source behaves as the states

$$|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$$

With probabilities given by

$$\langle v_1|D|v_1\rangle, \langle v_2|D|v_2\rangle, \ldots, \langle v_n|D|v_n\rangle.$$

The source has entropy given by

$$\boxed{-\sum_i \langle v_i|D|v_i\rangle \times \log_2\left(\langle v_i|D|v_i\rangle\right).}$$

Example 7.2.4: Use the information of Ex ~~6~~7.2.2. to compute the entropy $\{|0\rangle, |1\rangle\}$

$$D = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 1/3 \end{pmatrix}$$

$$\langle 0|D|0\rangle = \frac{2}{3}$$

$$\langle 1|D|1\rangle = 1/3$$

$$H_S = -\left[\langle 0|D|0\rangle \log_2(\langle 0|D|0\rangle) + \langle 1|D|1\rangle \log_2(\langle 1|D|1\rangle)\right]$$

$$H_s = -\left[\frac{2}{3}\log_2\left(\frac{2}{3}\right) + \frac{1}{3}\log_2\left(\frac{1}{3}\right)\right] = \underline{0.918}$$

Definition 7.2.2: The **von Neumann entropy** for a quantum source represented by a density operator D is given by

$$H_V = -\sum_{i=1}^{n} \lambda_i \log_2(\lambda_i)$$

where $\lambda_i's$ are the eigenvalues of D

$$D = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

Example 7.2.5: Find the Von Neumann entropy of the quantum state in example ~~9~~7.2.3.

$$\det(D - \lambda I) = 0.$$

$2(3 \times 4)$

$$\det\begin{bmatrix} \frac{5}{6}-\lambda & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6}-\lambda \end{bmatrix} = 0$$

$$\left(\frac{5}{6}-\lambda\right)\left(\frac{1}{6}-\lambda\right) - \frac{1}{36} = 0$$

$(36) \times$

$$\frac{1}{6}(5-6\lambda)\frac{1}{6}(1-6\lambda) - \frac{1}{36} = 0$$

$$(5-6\lambda)(1-6\lambda) - 1 = 0$$

$$36\lambda^2 - 36\lambda + 5 - 1 = 0$$

$(/4)$

$$36\lambda^2 - 36\lambda + 4 = 0$$

$$9\lambda^2 - 9\lambda + 1 = 0$$

$$a = 9, \quad b = -9, \quad c = 1$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{9 \pm \sqrt{81 - 36}}{18} = \frac{9 \pm 3\sqrt{5}}{18} = \frac{3 \pm \sqrt{5}}{6}$$

$$\lambda_1 = \frac{3+\sqrt{5}}{6} = 0.872 \qquad \lambda_2 = \frac{3-\sqrt{5}}{6} = 0.127$$

$$H_V = -\sum_i \lambda_i \log_2 \lambda_i = -\left[0.872\log_2(0.872) + 0.127\log_2(0.127)\right]$$

$$= 0.55$$

Example 7.2.6: Find the Von Neumann entropy of the density matrix

$$D = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \qquad \det(D - \lambda I) = 0$$

$$\begin{vmatrix} \frac{1}{2}-\lambda & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2}-\lambda \end{vmatrix} = 0 \Rightarrow \left(\frac{1}{2}-\lambda\right)^2 - \frac{1}{16} = 0.$$

$$\left(\frac{1}{2}-\lambda\right)^2 = \frac{1}{16}$$

$$\frac{1}{2}-\lambda = \pm\frac{1}{4}$$

$$\frac{1}{2}-\lambda_1 = \frac{1}{4} \qquad \frac{1}{2}-\lambda_2 = -1/4$$

$$\lambda_1 = 1/4 \qquad \lambda_2 = 3/4$$

$$H_V = -\left[\frac{1}{4}\log_2\frac{1}{4} + \frac{3}{4}\log_2(3/4)\right] = \underline{\underline{0.811}}$$