Question 1: Concept of Photon [6 marks]

A 500 W street lamp is at a distance to 1.0 km from and an observer. If observer's eye lens is of 5 mm in diameter find the number photons hitting his retina per second. (assume: lamp is producing light equally in all directions of wave length 600 nm, there is no absorption of light by atmosphere, and light obeys the inverse square law)

Solution:

Step 1: Calculate the intensity at distance $r = 1000$ m using the inverse square law:

$$I = \frac{P_{\text{source}}}{4\pi r^2} = \frac{500\ \text{W}}{4\pi(1000\ \text{m})^2} = \frac{500}{4\pi \times 10^6}\ \text{W/m}^2 \tag{1}$$

Step 2: Calculate the power incident on the eye lens (diameter $d = 5$ mm):

$$P_{\text{lens}} = I \times A_{\text{lens}} = I \times \pi \left(\frac{d}{2}\right)^2 \tag{2}$$

$$= \frac{500}{4\pi \times 10^6} \times \pi \left(\frac{5 \times 10^{-3}}{2}\right)^2\ \text{W} \tag{3}$$

Step 3: Calculate the energy per photon ($\lambda = 600$ nm):

$$E_{\text{photon}} = h\nu = \frac{hc}{\lambda} = \frac{(6.626 \times 10^{-34})(3 \times 10^8)}{600 \times 10^{-9}}\ \text{J} \tag{4}$$

Step 4: Calculate the number of photons per second:

$$N = \frac{P_{\text{lens}}}{E_{\text{photon}}} = \frac{P_{\text{lens}} \times \lambda}{hc}\ photons/s = \boxed{2.36 \times 10^9\ photons/s} \tag{5}$$

Question 2: Normalization [6 marks]

Verify each of the following state is normalized, if the state is NOT normalized, normalized the state (show all the steps of verification and normalization):

(i) $|\varphi\rangle = \frac{1}{\sqrt{6}}|0110\rangle + \frac{2}{\sqrt{3}}|1111\rangle + \frac{1}{\sqrt{6}}|0101\rangle$

Solution:

$$\langle\varphi|\varphi\rangle = \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\frac{2}{\sqrt{3}}\right|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 \tag{6}$$
$$= \frac{1}{6} + \frac{4}{3} + \frac{1}{6} \tag{7}$$
$$= \frac{1}{6} + \frac{8}{6} + \frac{1}{6} = \frac{10}{6} \neq 1 \tag{8}$$

State is **not normalized**. Normalized state:

$$|\varphi_{norm}\rangle = \sqrt{\frac{6}{10}}|\varphi\rangle = \boxed{\sqrt{\frac{3}{5}}|\varphi\rangle}$$

(ii) $|\Psi\rangle = \frac{1}{\sqrt{6}}|11\rangle + \frac{\sqrt{2}}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle$

Solution:

$$\langle\Psi|\Psi\rangle = \left|\frac{1}{\sqrt{6}}\right|^2 + \left|\frac{\sqrt{2}}{\sqrt{3}}\right|^2 + \left|\frac{1}{\sqrt{6}}\right|^2 \tag{9}$$
$$= \frac{1}{6} + \frac{2}{3} + \frac{1}{6} \tag{10}$$
$$= \frac{1}{6} + \frac{4}{6} + \frac{1}{6} = 1 \tag{11}$$

State is **normalized**.

(iii) $|\Upsilon\rangle = \frac{3i|0\rangle + 4|1\rangle}{5}$

Solution:

$$\langle\Upsilon|\Upsilon\rangle = \frac{1}{25}\left(|3i|^2 + |4|^2\right) \tag{12}$$
$$= \frac{1}{25}(9 + 16) \tag{13}$$
$$= \frac{25}{25} = 1 \tag{14}$$

State is **normalized**.

Question 3: Change of Basis [6 marks]

Quantum state $|+\rangle$ and $|-\rangle$ is given in terms of $|0\rangle$ and $|1\rangle$:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Express the states $|0\rangle$ and $|1\rangle$ in terms of $|+\rangle$ and $|-\rangle$.

Solution:

$$|+\rangle + |-\rangle = \frac{2}{\sqrt{2}}|0\rangle = \sqrt{2}|0\rangle \Rightarrow \boxed{|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle} \tag{15}$$

$$|+\rangle - |-\rangle = \frac{2}{\sqrt{2}}|1\rangle = \sqrt{2}|1\rangle \Rightarrow \boxed{|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle} \tag{16}$$

$$\tag{17}$$

Question 4: Probability of measuring a state [6 marks]

In each of the following quantum states:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$$

$$|\phi\rangle = \frac{1+i}{\sqrt{3}}|0\rangle - \sqrt{\frac{1}{3}}|1\rangle$$

(a) What is the probability that we find the qubit in state $|0\rangle$?

Solution:

$$|\langle 0|\psi\rangle|^2 = \left|\frac{1}{\sqrt{3}}\underbrace{\langle 0|0\rangle}_{1} + \sqrt{\frac{2}{3}}\underbrace{\cancel{\langle 0|1\rangle}}_{0}\right|^2 = \left|\frac{1}{\sqrt{3}}\right|^2 = \boxed{33.3\%} \tag{18}$$

$$|\langle 0|\phi\rangle|^2 = \left|\frac{1+i}{\sqrt{3}}\underbrace{\langle 0|0\rangle}_{1} - \sqrt{\frac{1}{3}}\underbrace{\cancel{\langle 0|1\rangle}}_{0}\right|^2 = \left|\frac{1+i}{\sqrt{3}}\right|^2 = \boxed{66.6\%} \tag{19}$$

$$\tag{20}$$

(b) What is the probability that we find the qubit in state $|1\rangle$?

Solution:

$$|\langle 1|\psi\rangle|^2 = \left|\frac{1}{\sqrt{3}}\underbrace{\cancel{\langle 1|0\rangle}}_{0} + \sqrt{\frac{2}{3}}\underbrace{\langle 1|1\rangle}_{1}\right|^2 = \left|\sqrt{\frac{2}{3}}\right|^2 = \boxed{66.6\%} \tag{21}$$

$$|\langle 1|\phi\rangle|^2 = \left|\frac{1+i}{\sqrt{3}}\underbrace{\cancel{\langle 1|0\rangle}}_{0} - \sqrt{\frac{1}{3}}\underbrace{\langle 1|1\rangle}_{1}\right|^2 = \left|-\sqrt{\frac{1}{3}}\right|^2 = \boxed{33.3\%} \tag{22}$$

$$\tag{23}$$

Question 5: Entangled States [5 marks]

For given entangled state $|\psi\rangle$:

$$|\psi\rangle = \left(\frac{1}{4} + \frac{i}{4}\right)|01\rangle + \frac{\sqrt{7}}{2\sqrt{2}}|10\rangle$$

(a) What is the probability of finding the first qubit in state $|1\rangle$?

Solution:
Since the first qubit is $|1\rangle$ only in the $|10\rangle$ term:

$$P(\text{first qubit} = 1) = \left|\frac{\sqrt{7}}{2\sqrt{2}}\right|^2 = \frac{7}{8} = \boxed{87.5\%} \tag{24}$$

(b) If you measure the first qubit in state $|1\rangle$, what is/are the possible states of second qubit?

Solution:
Since the first qubit $|1\rangle$ only appears in the $|10\rangle$ term, the second qubit must be:

$$\boxed{|0\rangle} \tag{25}$$

(c) Replace the factor $\left(\frac{1}{4} + \frac{i}{4}\right)$ with new number (fraction/real/complex) in state $|\psi\rangle$ such the probability of finding the state $|01\rangle$ remains the same.

Solution:

$$\left|\left(\frac{1}{4} + \frac{i}{4}\right)\right|^2 = \frac{1}{8} = \left|\frac{1}{\sqrt{8}}\right|^2 \implies \boxed{|\psi\rangle = \frac{1}{\sqrt{8}}|01\rangle + \frac{\sqrt{7}}{2\sqrt{2}}|10\rangle} \tag{26}$$

$$\tag{27}$$