PHYS 512

Assignment – 03 $\langle \mathit{Quantum} | \mathit{Computing} \rangle$

Due Sunday Oct 04 – 11:59 pm (NO late submission)

Solution

Q.1 Ket and Bra, Orthonormality

Consider the following two kets:

$$|\psi\rangle = \begin{pmatrix} -3i \\ 2+i \\ 4 \end{pmatrix}, \qquad |\varphi\rangle = \begin{pmatrix} 2 \\ -i \\ 2-3i \end{pmatrix}$$

(i) Find the bra $\langle\varphi|$

(ii) Find $(|\psi\rangle)^*$

(iii) Are $|\psi\rangle$ and $|\varphi\rangle$ are orthogonal?

(iv) Briefly explain, why the product $|\psi\rangle|\varphi\rangle$ *and* $\langle\varphi|\langle\psi|$ do not make sense?

Solution

(i)

$$\langle\varphi| = (2 \quad i \quad 2+3i)$$

(ii)

$$(|\psi\rangle)^\dagger = \langle\psi| = (3i \quad 2-i \quad 4)$$

(iii)

$$\langle\varphi|\psi\rangle = (2 \quad i \quad 2+3i)\begin{pmatrix} -3i \\ 2+i \\ 4 \end{pmatrix} = 7+8i \neq 0,$$ *Therefore given states are not orthogonal*

(iV) *Such matrix multiplication is not possible*

Q.2 Ket, Bra and inner product

Consider following quantum states

$$|\psi\rangle = 3i|\varphi_1\rangle - 7i|\varphi_2\rangle, \qquad |\chi\rangle = -|\varphi_1\rangle + 2i|\varphi_2\rangle$$

Where $|\varphi_1\rangle$ *and* $|\varphi_2\rangle$ are orthonormal. Calculate the inner (scaler) product $\langle\psi|\chi\rangle$ $\langle\chi|\psi\rangle$. Are they equal?

Solution

$$\langle\psi|\chi\rangle = (\langle\varphi_1|(-3i) + \langle\varphi_2|7i)[-|\varphi_1\rangle + 2i|\varphi_2\rangle] = -14+3i$$

$$\langle\chi|\varphi\rangle = (\langle\varphi_1| + \langle\varphi_2|(-2i))[3i|\varphi_1\rangle - 7i|\varphi_2\rangle] = -14-3i$$

Q.3 Hermitian

Check the hermicity of operators $(\hat{A}+\hat{A}^{\dagger}),\ i(\hat{A}+\hat{A}^{\dagger}),\ i(\hat{A}-\hat{A}^{\dagger}),$

Solution

$(\hat{A}+\hat{A}^{\dagger})^{\dagger} = (\hat{A}+\hat{A}^{\dagger})$ *Therefore it is Hermitian*

$[i(\hat{A}+\hat{A}^{\dagger})]^{\dagger} = -\,i(\hat{A}+\hat{A}^{\dagger})$ *NOT Hermitian*

$[i(\hat{A}-\hat{A}^{\dagger})]^{\dagger} = -i(-\hat{A}+\hat{A}^{\dagger}) = \ i(\hat{A}-\hat{A}^{\dagger})$ *Therefore it is Hermitian*

Q.4 Outer product

Consider the two States

$$|\psi\rangle = i|\varphi_1\rangle + 3i|\varphi_2\rangle - |\varphi_3\rangle, \qquad |\chi\rangle = |\varphi_1\rangle - i|\varphi_2\rangle + 5i|\varphi_3\rangle$$

Where $|\varphi_1\rangle, |\varphi_2\rangle, and\ |\varphi_3\rangle$ are orthogonal and normalized.

(i) Calculate $|\psi\rangle\langle\chi|\ and\ |\chi\rangle\langle\psi|$. Are they equal?

(ii) Find Hermitian conjugate of $|\psi\rangle\langle\chi|$

Solution

$$|\psi\rangle\langle\chi| = \begin{pmatrix} 1 \\ -i \\ 5i \end{pmatrix}\begin{pmatrix} -i & -3i & -1 \end{pmatrix} = \begin{pmatrix} i & -1 & 5 \\ 3i & -3 & 15 \\ -1 & -i & 5i \end{pmatrix}$$

$$|\chi\rangle\langle\psi| = \begin{pmatrix} -i \\ -3i \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -i & 5i \end{pmatrix} = \begin{pmatrix} -i & -3i & -1 \\ -1 & -3 & i \\ 5 & 15 & -5i \end{pmatrix}$$

$$(|\psi\rangle\langle\chi|)^{\dagger} = |\chi\rangle\langle\psi| = \begin{pmatrix} -i & -3i & -1 \\ -1 & -3 & i \\ 5 & 15 & -5i \end{pmatrix}$$

Q.5 Hermitian

Consider the following operator:

$$\hat{A} = \begin{pmatrix} 0 & 0 & \frac{2}{1+i} \\ 0 & 0 & 0 \\ 1+i & 0 & 0 \end{pmatrix},$$

(v) Operator $\hat{A}$ is Hermitian?

(vi) Operator $\hat{A}$ is unitary?

(Clearly show all the working for Hermitian and unitary)

Solution

(i)

$$\hat{A} = \begin{pmatrix} 0 & 0 & \frac{2}{1+i} \\ 0 & 0 & 0 \\ 1+i & 0 & 0 \end{pmatrix} \Rightarrow \hat{A} = \begin{pmatrix} 0 & 0 & 1-i \\ 0 & 0 & 0 \\ 1+i & 0 & 0 \end{pmatrix} \text{ now clearly A is Hermitian}$$

(ii) $\hat{A}\hat{A}^\dagger = \begin{pmatrix} 0 & 0 & 1-i \\ 0 & 0 & 0 \\ 1+i & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1-i \\ 0 & 0 & 0 \\ 1+i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-i^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1-i^2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Not an Identity matrix, therefore A is NOT unitary

Q.6 Cauchy-Schwartz and Triangle inequality

Consider the following two states:

$$|\psi\rangle = -3|0\rangle - 2i|1\rangle, \qquad |\phi\rangle = |0\rangle + 5|1\rangle$$

Show that these two states obeys

(i) Cauchy-Schwartz inequality: $|\langle\psi||\phi\rangle|^2 \le \langle\psi|\psi\rangle\langle\phi||\phi\rangle$

(ii) Triangle inequality: $\sqrt{\langle(\psi+\phi)|\psi+\phi\rangle} \le \sqrt{\langle\psi||\psi\rangle} + \sqrt{\langle\phi||\phi\rangle}$

Solution

$$\langle\psi|\varphi\rangle = (\langle 0|(-3) + \langle 1|2i)(-3|0\rangle - 2i|1\rangle) = -3 + 10i$$

$$\langle\psi|\psi\rangle = 13$$

$$\langle\varphi|\varphi\rangle = 26$$

$$\langle(\psi+\varphi)|(\psi+\varphi)\rangle = (\langle 0|(-2) + \langle 1|(5+2i))(-2|0\rangle + (5-2i)|1\rangle) = (-2)^2 + (5+2i)(5-2i) = 33$$

Q.7 Find new states

Consider following three quantum states

$$|\psi_1\rangle = |0\rangle, \qquad |\psi_2\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, \qquad |\psi_3\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Find new states $|\tilde{\psi}_1\rangle, |\tilde{\psi}_2\rangle, and\ |\tilde{\psi}_3\rangle$ that are normalized, superposition states of $|0\rangle\ and\ |1\rangle$ and orthogonal to $|\psi_1\rangle, |\psi_2\rangle, and\ |\psi_3\rangle$, respectively.

Solution

Let

$$|\tilde{\psi}_1\rangle = a|0\rangle + b|1\rangle, \text{ where } a^2 + b^2 = 1 \text{ (new states is normalized)}$$

$$|\tilde{\psi}_2\rangle = c|0\rangle + d|1\rangle, \text{where } c^2 + d^2 = 1 \text{ (new state is normalized)}$$
$$|\tilde{\psi}_3\rangle = e|0\rangle + f|1\rangle, \text{where } e^2 + f^2 = 1 \text{ (new state is normalized)}$$

Using above normalization and following orthogonal conditions

$\langle\psi_1|\tilde{\psi}_1\rangle = 0 \quad \langle\psi_2|\tilde{\psi}_2\rangle = 0 \quad \langle\psi_3|\tilde{\psi}_3\rangle = 0$

To find the values of a, b, c, d, e, and f

Q.8 Direct Product/ combining quantum states

Show that the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle,$$

Can be expressed in-terms of $|+\,+\rangle$ *and* $|-\,-\rangle$ basis, i.e.

$$|\psi\rangle = \frac{1}{\sqrt{2}}|+\,+\rangle + \frac{1}{\sqrt{2}}|-\,-\rangle,$$

Solution

$$|00\rangle = |0\rangle\otimes|0\rangle = \frac{1}{2}[(|+\rangle + |-\rangle)\otimes(|+\rangle + |-\rangle)] = \frac{1}{2}[|+\,+\rangle + |+\,-\rangle + |-\,+\rangle + |-\,-\rangle]$$

Similarly

$$|11\rangle = |1\rangle\otimes|1\rangle = \frac{1}{2}[(|+\rangle - |-\rangle)\otimes(|+\rangle - |-\rangle)] = \frac{1}{2}[|+\,+\rangle - |+\,-\rangle - |-\,+\rangle + |-\,-\rangle]$$

Therefore

$$|00\rangle + |11\rangle = [|+\,+\rangle + |-\,-\rangle]$$
$$\Rightarrow |\psi\rangle = \frac{1}{\sqrt{2}}|+\,+\rangle + \frac{1}{\sqrt{2}}|-\,-\rangle$$

Q.9 Expectation value

Consider the a states which is given in terms of three orthonormal vectors $|\varphi_1\rangle, |\varphi_2\rangle, and\ |\varphi_3\rangle$ as follows

$$|\psi\rangle = \frac{1}{\sqrt{15}}|\varphi_1\rangle + \frac{1}{\sqrt{3}}|\varphi_2\rangle + \frac{1}{\sqrt{5}}|\varphi_3\rangle$$

Where $|\varphi_n\rangle$ are eigenstates to an operator $\hat{B}$ such that $\hat{B}|\varphi_n\rangle = (3n^2 - 1)|\varphi_n\rangle\ with\ n = 1,2,3$

(i) Find the norm of the state $|\psi\rangle$.

(ii) Find the expectation value of $\hat{B}$ for the state $|\psi\rangle$.

(iii) Find the expectation value of $\hat{B}^2$ for the state $|\psi\rangle$.

Solution

(i) Norm of state: …….(you did it many times)

(ii) Expectation value of operator B

$$\langle\psi|\hat{B}|\psi\rangle = \left(\left\langle\varphi_1\right|\frac{1}{\sqrt{15}} + \langle\varphi_2|\frac{1}{\sqrt{3}} + \langle\varphi_3|\frac{1}{\sqrt{5}}\right) + \hat{B}\left(\frac{1}{\sqrt{15}}|\varphi_1\rangle + \frac{1}{\sqrt{3}}|\varphi_2\rangle + \frac{1}{\sqrt{5}}|\varphi_3\rangle\right)$$

$$\langle\psi|\hat{B}|\psi\rangle = \left(\left\langle\varphi_1\right|\frac{1}{\sqrt{15}} + \langle\varphi_2|\frac{1}{\sqrt{3}} + \langle\varphi_3|\frac{1}{\sqrt{5}}\right) + \left(\frac{1}{\sqrt{15}}\hat{B}|\varphi_1\rangle + \frac{1}{\sqrt{3}}\hat{B}|\varphi_2\rangle + \frac{1}{\sqrt{5}}\hat{B}|\varphi_3\rangle\right)$$

$$\langle\psi|\hat{B}|\psi\rangle = \left(\left\langle\varphi_1\right|\frac{1}{\sqrt{15}} + \langle\varphi_2|\frac{1}{\sqrt{3}} + \langle\varphi_3|\frac{1}{\sqrt{5}}\right) + \left(\frac{2}{\sqrt{15}}|\varphi_1\rangle + \frac{11}{\sqrt{3}}|\varphi_2\rangle + \frac{26}{\sqrt{5}}|\varphi_3\rangle\right)$$

$$\langle\psi|\hat{B}|\psi\rangle = \left(\frac{2}{15} + \frac{11}{3} + \frac{26}{5}\right) = 9$$

(iii) Similarly you can find the expectation value of operator B^2

Q.10 Define the mixed expression

In the following expression, where is an operator, specify the nature of each expression (i.e. specify whether it is an operator, a ket, or a bra); then find it Hermitian conjugate.

(i) $\langle\psi|\hat{A}|\psi\rangle\langle\psi|$
(ii) $\hat{A}|\psi\rangle\langle\varphi|$
(iii) $(|\varphi\rangle\langle\varphi|\hat{A}) - i\left(\hat{A}|\psi\rangle\langle\psi|\right)$

Solution

(i) **Bra**
(ii) **Operator**
(iii) **Operator**

To find the Hermitian of each expression, use the following rule(s)

$$[ABC]^\dagger = C^\dagger B^\dagger A^\dagger$$

$$\left[\hat{A}|\psi\rangle\right]^\dagger = \langle\psi|\hat{A}^\dagger$$

$$[\langle\psi|\varphi\rangle]^\dagger = \langle\varphi|\psi\rangle$$

Q.11 Combining quantum states

Consider following two quantum states in computational basis

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big|0\rangle - \frac{i}{\sqrt{2}}\Big|1\rangle, \qquad |\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

Compute the following

(i) $|\psi\rangle \otimes |\varphi\rangle$
(ii) $\langle\psi| \otimes \langle\varphi|$
(iii) $(\sigma_x \otimes I)(|\psi\rangle \otimes |\varphi\rangle)$
(iv) $(\sigma_x \otimes \sigma_x)(|\psi\rangle \otimes |\varphi\rangle)$

<u>**Solution**</u>

(i) $$|\psi\rangle \otimes |\varphi\rangle = \left[\frac{1}{\sqrt{2}}\Big|0\rangle - \frac{i}{\sqrt{2}}\Big|1\rangle\right] \otimes \left[\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle\right]$$

$$|\psi\rangle \otimes |\varphi\rangle = \frac{1}{2}\Big|00\rangle + \frac{i}{2}\Big|01\rangle - \frac{i}{2}|10\rangle + \frac{1}{2}|11\rangle$$

(ii)

Similarly you can find $\langle\psi| \otimes \langle\varphi|$

(iii) $(\sigma_x \otimes I)(|\psi\rangle \otimes |\varphi\rangle)$

$$\sigma_x \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv (4x4\ matrix) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$|\psi\rangle \otimes |\varphi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \equiv (4x1\ matrix) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Xply equation (1) with (2), to get the final answer

(iv) Like above you can solve this part as well