Question 1

Ket and Bra, Orthonormality

Consider the following two kets:

$$|\psi\rangle = \begin{pmatrix} -3i \\ 2+i \\ 4 \end{pmatrix}, \quad |\varphi\rangle = \begin{pmatrix} 2 \\ -i \\ 2-3i \end{pmatrix}$$

(i) Find the bra $\langle\varphi|$

(ii) Find $(|\psi\rangle)^*$

(iii) Are $|\psi\rangle$ and $|\varphi\rangle$ orthogonal?

(iv) Briefly explain, why the product $|\psi\rangle|\varphi\rangle$ and $\langle\varphi|\langle\psi|$ do not make sense?

Solution:

(i) $\langle\varphi| = \begin{pmatrix} 2 & i & 2+3i \end{pmatrix}$

(ii) $(|\psi\rangle)^* = \begin{pmatrix} 3i \\ 2-i \\ 4 \end{pmatrix}$

(iii)

$$\begin{aligned} \langle\varphi|\psi\rangle &= \begin{pmatrix} 2 & i & 2+3i \end{pmatrix} \begin{pmatrix} -3i \\ 2+i \\ 4 \end{pmatrix} \\ &= 2(-3i) + i(2+i) + (2+3i)(4) \\ &= -6i + 2i - 1 + 8 + 12i = 7 + 8i \neq 0 \end{aligned}$$

$\therefore$ not orthogonal.

(iv) $|\psi\rangle$ is 3×1, $|\varphi\rangle$ is 3×1. The product $(3\times 1)(3\times 1)$ is undefined. Similarly, $(1\times 3)(1\times 3)$ is undefined. However, people do use similar notation $|\psi\rangle|\varphi\rangle$ to resemble the tensor product $|\psi\rangle\otimes|\varphi\rangle$, which is well-defined as the direct product (Kronecker product) of the two state vectors, resulting in a 9×1 vector in this case.

Question 2

Ket, Bra and inner product

Consider following quantum states

$$|\psi\rangle = 3i|\varphi_1\rangle - 7i|\varphi_2\rangle, \quad |\chi\rangle = -|\varphi_1\rangle + 2i|\varphi_2\rangle$$

Where $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are orthonormal. Calculate the inner (scalar) product $\langle\psi|\chi\rangle$ $\langle\chi|\psi\rangle$. Are they equal?

Solution:

$$\begin{aligned} \langle\psi|\chi\rangle &= (-3i)(-1) + (7i)(2i) = 3i - 14 \\ \langle\chi|\psi\rangle &= (-1)(3i) + (-2i)(-7i) = -3i - 14 \end{aligned}$$

Not equal. In fact, $(\langle\chi|\psi\rangle)^* = \langle\psi|\chi\rangle$.

Question 3

Hermitian

Check the hermiticity of operators $(\hat{A} + \hat{A}^\dagger)$, $i(\hat{A} + \hat{A}^\dagger)$, $i(\hat{A} - \hat{A}^\dagger)$.

Solution:

$$\begin{aligned}
(\hat{A} + \hat{A}^\dagger)^\dagger &= \hat{A}^\dagger + \hat{A} = \hat{A} + \hat{A}^\dagger \quad \text{Hermitian} \\
[i(\hat{A} + \hat{A}^\dagger)]^\dagger &= -i(\hat{A}^\dagger + \hat{A}) \neq i(\hat{A} + \hat{A}^\dagger) \quad \text{Not Hermitian} \\
[i(\hat{A} - \hat{A}^\dagger)]^\dagger &= -i(\hat{A}^\dagger - \hat{A}) = i(\hat{A} - \hat{A}^\dagger) \quad \text{Hermitian}
\end{aligned}$$

Question 4

Outer product

Consider the two States

$$|\psi\rangle = i|\varphi_1\rangle + 3i|\varphi_2\rangle - |\varphi_3\rangle, \quad |\chi\rangle = |\varphi_1\rangle - i|\varphi_2\rangle + 5i|\varphi_3\rangle$$

Where $|\varphi_1\rangle$, $|\varphi_2\rangle$, and $|\varphi_3\rangle$ are orthogonal and normalized.

(i) Calculate $|\psi\rangle\langle\chi|$ and $|\chi\rangle\langle\psi|$. Are they equal?

(ii) Find Hermitian conjugate of $|\psi\rangle\langle\chi|$

Solution:

(i)

$$\begin{aligned}
|\psi\rangle\langle\chi| &= \begin{pmatrix} i \\ 3i \\ -1 \end{pmatrix} \begin{pmatrix} 1 & i & -5i \end{pmatrix} = \begin{pmatrix} i & -1 & 5 \\ 3i & -3 & 15 \\ -1 & -i & 5i \end{pmatrix} \\
|\chi\rangle\langle\psi| &= \begin{pmatrix} 1 \\ -i \\ 5i \end{pmatrix} \begin{pmatrix} -i & -3i & -1 \end{pmatrix} = \begin{pmatrix} -i & -3i & -1 \\ -1 & -3 & i \\ 5 & 15 & -5i \end{pmatrix}
\end{aligned}$$

Not equal.

(ii) $(|\psi\rangle\langle\chi|)^\dagger = |\chi\rangle\langle\psi| = \begin{pmatrix} -i & -3i & -1 \\ -1 & -3 & i \\ 5 & 15 & -5i \end{pmatrix}$

Question 5

Hermitian

Consider the following operator:

$$\hat{A} = \begin{pmatrix} 0 & 0 & \frac{2}{1+i} \\ 0 & 0 & 0 \\ 1+i & 0 & 0 \end{pmatrix}$$

(v) Operator $\hat{A}$ is Hermitian?

(vi) Operator $\hat{A}$ is unitary?

(Clearly show all the working for Hermitian and unitary)

Solution:

(v) Hermitian. Since:

$$\hat{A}^\dagger = \overline{\hat{A}}^T = \begin{pmatrix} 0 & 0 & 1+i \\ 0 & 0 & 0 \\ 1-i & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 1-i \\ 0 & 0 & 0 \\ 1+i & 0 & 0 \end{pmatrix} = \hat{A}$$

Note $\frac{2}{1+i} = \frac{2(1-i)}{2} = 1-i$.

(vi) Not Unitary. Since:

$$\hat{A}\hat{A}^\dagger = \begin{pmatrix} 0 & 0 & 1-i \\ 0 & 0 & 0 \\ 1+i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1+i \\ 0 & 0 & 0 \\ 1-i & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \neq I$$

Question 6

Cauchy-Schwartz and Triangle inequality

Consider the following two states:

$$|\psi\rangle = -3|0\rangle - 2i|1\rangle, \quad |\phi\rangle = |0\rangle + 5|1\rangle$$

Show that these two states obeys

(i) Cauchy-Schwartz inequality: $|\langle\psi|\phi\rangle|^2 \leqslant \langle\psi|\psi\rangle\langle\phi|\phi\rangle$

(ii) Triangle inequality: $\sqrt{\langle(\psi+\phi)|\psi+\phi\rangle} \leqslant \sqrt{\langle\psi|\psi\rangle} + \sqrt{\langle\phi|\phi\rangle}$

Solution:

(i)

$$\langle\psi|\psi\rangle = 9 + 4 = 13, \quad \langle\phi|\phi\rangle = 1 + 25 = 26$$
$$|\langle\psi|\phi\rangle|^2 = |-3 + 10i|^2 = 109 \leqslant 338 = 13 \times 26 \quad \checkmark$$

(ii)

$$\langle(\psi+\phi)|(\psi+\phi)\rangle = |-2|^2 + |5-2i|^2 = 4 + 29 = 33$$
$$\sqrt{33} \approx 5.74 \leqslant \sqrt{13} + \sqrt{26} \approx 8.71 \quad \checkmark$$

Question 7

Find new states

Consider following three quantum states

$$|\psi_1\rangle = |0\rangle, \quad |\psi_2\rangle = -\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle, \quad |\psi_3\rangle = -\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

Find new states $|\tilde{\psi}_1\rangle$, $|\tilde{\psi}_2\rangle$, and $|\tilde{\psi}_3\rangle$ that are normalized, superposition states of $|0\rangle$ and $|1\rangle$ and orthogonal to $|\psi_1\rangle$, $|\psi_2\rangle$, and $|\psi_3\rangle$, respectively.

Solution:

$$\begin{aligned}
|\tilde{\psi}_1\rangle &= |1\rangle, \quad \langle\psi_1|\tilde{\psi}_1\rangle = 0 \quad \checkmark \\
|\tilde{\psi}_2\rangle &= -\frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle, \quad \langle\psi_2|\tilde{\psi}_2\rangle = \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = 0 \quad \checkmark \\
|\tilde{\psi}_3\rangle &= \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle, \quad \langle\psi_3|\tilde{\psi}_3\rangle = -\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} = 0 \quad \checkmark
\end{aligned}$$

Question 8

Direct Product/ combining quantum states

Show that the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle,$$

Can be expressed in-terms of $|++\rangle$ and $|--\rangle$ basis, i.e.

$$|\psi\rangle = \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle,$$

Solution:

Using $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$:

$$|00\rangle = \frac{1}{2}(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle)$$
$$|11\rangle = \frac{1}{2}(|++\rangle - |+-\rangle - |-+\rangle + |--\rangle)$$

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
&= \frac{1}{2\sqrt{2}}(2|++\rangle + 2|--\rangle) = \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle
\end{aligned}$$

Question 9

Expectation value

Consider the a states which is given in terms of three orthonormal vectors $|\varphi_1\rangle$, $|\varphi_2\rangle$, and $|\varphi_3\rangle$ as follows

$$|\psi\rangle = \frac{1}{\sqrt{15}}|\varphi_1\rangle + \frac{1}{\sqrt{3}}|\varphi_2\rangle + \frac{1}{\sqrt{5}}|\varphi_3\rangle$$

Where $|\varphi_n\rangle$ are eigenstates to an operator $\hat{B}$ such that $\hat{B}|\varphi_n\rangle = (3n^2 - 1)|\varphi_n\rangle$ with $n = 1, 2, 3$

(iii) Find the norm of the state $|\psi\rangle$.

(iv) Find the expectation value of $\hat{B}$ for the state $|\psi\rangle$.

(v) Find the expectation value of $\hat{B}^2$ for the state $|\psi\rangle$.

Solution:

Eigenvalues: $\lambda_1 = 2, \lambda_2 = 11, \lambda_3 = 26$. Matrix: $\hat{B} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 26 \end{pmatrix}$

(iii)

$$\langle\psi|\psi\rangle = \frac{1}{15} + \frac{1}{3} + \frac{1}{5} = \frac{3}{5}, \quad \|\psi\| = \sqrt{\frac{3}{5}}$$

(iv)

$$\langle\hat{B}\rangle = \frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\frac{1}{15}(2) + \frac{1}{3}(11) + \frac{1}{5}(26)}{\frac{3}{5}} = \frac{9}{\frac{3}{5}} = 15$$

(v)

$$\langle\hat{B}^2\rangle = \frac{\langle\psi|\hat{B}^2|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\frac{1}{15}(4) + \frac{1}{3}(121) + \frac{1}{5}(676)}{\frac{3}{5}} = \frac{\frac{879}{5}}{\frac{3}{5}} = \frac{879}{3} = 293$$

Question 10

Define the mixed expression

In the following expression, where $\hat{A}$ is an operator, specify the nature of each expression (i.e. specify whether it is an operator, a ket, or a bra); then find it Hermitian conjugate.

(i) $\langle\psi|\hat{A}|\psi\rangle\langle\psi|$

(ii) $\hat{A}|\psi\rangle\langle\varphi|$

(iii) $(|\varphi\rangle\langle\varphi|\hat{A}) - i(\hat{A}|\psi\rangle\langle\psi|)$

Solution:

(i) Bra. Hermitian conjugate: $|\psi\rangle\langle\psi|\hat{A}^\dagger|\psi\rangle$

$$\underbrace{\langle\psi|\hat{A}|\psi\rangle}_{\text{scalar}}\underbrace{\langle\psi|}_{\text{bra}} = \text{scalar} \times \text{bra} = \text{bra}$$

(ii) Operator. Hermitian conjugate: $|\varphi\rangle\langle\psi|\hat{A}^\dagger$

$$\underbrace{\hat{A}|\psi\rangle}_{\text{ket}}\underbrace{\langle\varphi|}_{\text{bra}} = \text{ket} \times \text{bra} = \text{operator}$$

(iii) Operator. Hermitian conjugate: $\hat{A}^\dagger|\varphi\rangle\langle\varphi| + i|\psi\rangle\langle\psi|\hat{A}^\dagger$

$$\underbrace{(|\varphi\rangle\langle\varphi|\hat{A})}_{\text{operator}} - i\underbrace{(\hat{A}|\psi\rangle\langle\psi|)}_{\text{operator}} = \text{operator} - \text{operator} = \text{operator}$$

Question 11

Combining quantum states

Consider following two quantum states in computational basis

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle, \quad |\varphi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

Compute the following

(i) $|\psi\rangle \otimes |\varphi\rangle$

(ii) $\langle\psi| \otimes \langle\varphi|$

(iii) $(\sigma_x \otimes I)(|\psi\rangle \otimes |\varphi\rangle)$

(iv) $(\sigma_x \otimes \sigma_x)(|\psi\rangle \otimes |\varphi\rangle)$

Solution:

(i)

$$|\psi\rangle \otimes |\varphi\rangle = \frac{1}{2}(|00\rangle + i|01\rangle - i|10\rangle + |11\rangle) = \frac{1}{2}\begin{pmatrix} 1 \\ i \\ -i \\ 1 \end{pmatrix}$$

(ii)

$$\langle\psi| \otimes \langle\varphi| = \frac{1}{2}(\langle 00| - i\langle 01| + i\langle 10| + \langle 11|) = \frac{1}{2}\begin{pmatrix} 1 & -i & i & 1 \end{pmatrix}$$

(iii)

$$\sigma_x \otimes I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$(\sigma_x \otimes I)(|\psi\rangle \otimes |\varphi\rangle) = \frac{1}{2}\begin{pmatrix} -i \\ 1 \\ 1 \\ i \end{pmatrix}$$

(iv)

$$\sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$(\sigma_x \otimes \sigma_x)(|\psi\rangle \otimes |\varphi\rangle) = \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}$$